大韓數學會誌 數學

A NOTE ON THE ORDER IN B* ALGEBRAS

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INTRODUCTION

In the present paper the author made an attemption to apply properties of ordered linear spaces to complex B* algebras with identity. Consequent results are collected in Chapter II.

In his research the author found out that the set of those positive functionals giving irreducible *-representations for B* algebras (denoted by Ext P in this paper) is a good bridge connecting ordered spaces with B* algebras.

In Theorem 2.1 an extension, in some aspect, of the Gelfand-Naimark representation to a non-commutative B* algebra is made. Theorem 2.2 originally motivated by Theorem 1.2 characterizes those complex *-algebras whose hermitian subspaces with the typical positive cones are M spaces with order unit under suitable norms. In Theorem 2.3 the Kadison's problem (cf. (3)) is dealt with in the case of commutative complex B* algebras with identity.

The author interposed Chapter I to set up the notations and terminologies to be used throughout and some basic known results. All the other unexplained notations and terminologies will be found in (1) and (2). Notice that the definitions of an ordered Banach space and a GM space are slightly modified from the original ones.

CHAPTER I. PRELIMINARIES

Let (X, K) be an ordered linear space where X is a real linear space and K is a positive cone in X. We denote by K^* the set of all positive functionals on X and by cl K the cloure of K in the strongest locally convex topology for X.

If (X, K) is an ordered linear space such that cl K$\cap$(−cl K)={0} and K has an internal point e, then

$$\|x\| = \inf\{t>0: -te \leq x \leq te\}$$

is a norm in X. The norm $\|x\|$ is monotone in the sense that for any elements x and y of K $\|x\| \geq \|y\|$ if $x \geq y$. $\|x\|$ can also be defined by the equation

$$\|x\| = \sup\{|f(x)|: f \in K^*, f(e)=1\}.$$

The norm $\|x\|$ is called a NORM INDUCED BY e.

An ordered Banach space (X, K, $\|x\|$) is an ordered linear space (X, K) with a complete norm $\|x\|$.

Definition. An ordered Banach space (X, K, $\|x\|$) is a GM SPACE if and only if cl K$\cap$(−cl K)={0} and the norm $\|x\|$ can be induced by an internal point of K.

Let (X, K, $\|x\|$) be an ordered Banach space. The largest element in the unit sphere of X, if such exists, is called an ORDER UNIT. Remark that for a GM space the internal point of K inducing the GM space norm is a unique order unit.

Let (X, K, $\|x\|$) be a GM space with order unit e. For any f in K^* f is bounded and $\|f\| = f(e)$. We denote by P the subset of the adjoint space of X defined as

$$P = \{f \in K^*: f(e) \leq 1\}.$$

P is a compact convex subset of the adjoint space of X in the weak X topology. We denote by Ext P the set of nonzoro extremal points of P with the weak X topology.

If an ordered linear space (X, K) is a vector lattice, the absolute value of x in X shall be denoted by $|x|$.

Let (X_i, K_i) $(i=1, 2)$ be vector lattices. A linear map θ of X_1 into X_2 is called a LATTICE HOMOMORPHISM if $\theta(x\vee y)=\theta(x)\vee\theta(y)$. In the case X_2 is the lattice of real numbers, θ is called a LATTICE FUNCTIONAL.

Definition. An ordered Banach space (X, K, $\|x\|$) is an M SPACE if (X, K) is a vector lattice, $\|x\|$ is a lattice norm (*i. e.*, for any elements x and y of X $\|x\|\geq\|y\|$ if $|x|\geq|y|$) and $\|x\vee y\|=\|x\|\vee\|y\|$ for any elements x and y of K.

Let S be a topological space. Let C(S) he the algebra of all bounded complex valued continuous functions on S and C*(S) be the corresponding algebra of real valued functions. These are Banach algebras under the norm $\|f\|=\sup\{|f(w)|:w\in S\}$. We can order C*(S) by taking the positive cone to be the set of all functions which are everywhere nonnegative. If we are concerned with those algebras, the conjugate involution and previously defined norm and order will be given implicitly.

Let A be a complex *-algebra. Let H be a real linear space of hermitian elements of A. Let K_0 be the cone in H consisting of all elements that can be expressed as a finite sum of the form $\Sigma x_i^* x_i$. We denote by K the set cl K_0. We make the usual confusion between real valued functionals on H and complex valued functionals on A which are real on H.

Definition. If (X_i, K_i) $(i=1, 2)$ are ordered linear spaces, a linear isomorphism θ of X_1 into X_2 is an ORDER ISOMORPHISM if and only if $\theta(K_1)=\theta(X_1)\cap K_2$. If $A_i (i=1, 2)$ are complex B* algebras and (H_i, K_i) are the corresponding ordered linear spaces of hermitian elements, a linear isomorphism θ of A_1 into A_2 is an ORDER ISOMORPHISM if and only if θ is an order isomorphism of (H_1, K_1) into (H_2, K_2).

Theorem 1. 1. (Kakutani) Each M space (X, K, $\|x\|$) with order unit is isometric lattice-isomorphic, under evaluation, onto C*(S) where S is the set of all lattice functionals of norm one.

Theorem 1. 2. (P. E. Miles) Let A be a complex *-algebra with identity e. Then A admits a complete B* norm if and only if (H, K) is a GM space under the norm induced by e. In this case the GM space norm coincides with the B* norm restricted on H.

CHAPTER II. THEOREMS ON THE ORDER IN B* ALGEBRAS

Theorem 2. 1. Let A be a complex B* algebra with identity e. Then A is isometric order-isomorphic into C(Ext P). If A is commutative, then A is isometric *-isomorphic onto C(Ext P).

Proof. Notice that Ext P is well defined by Theorem 1.2.

If A is commutative, then Ext P is the carrier space of A——see, *e.g.*, (1; 4). The Gelfand-Naimark representation of A is the required isometric *-isomorphism.

Remark that the proof of the first part of the Theorem will be completed if we show that H is isometric order-isomorphic into C*(Ext P).

Let $\theta: x\rightarrow\hat{x}$ be the evaluation map of H into C*(Ext P). then $\|\hat{x}\|=\sup\{|f(x)|:f\in \text{Ext P}\}$ and by Theorem 1.2 $\|\hat{x}\|=\sup\{|f(x)|:f\in K^*, f(e)=1\}$. Since for any element f in Ext P $f(e)=\|f\|=1$, $\|\hat{x}\|\leq\|x\|$. Conversely let f be any positive functional such that $f(e)=1$. Then f is a member of P. By the Klein-Milman Theorem for any x of H and positive real number ε there exist positive real numbers $\alpha_1, \alpha_2, \cdots\cdots, \alpha_n$, and extremal points of P $f_1, f_2, \cdots\cdots, f_n$ such that $\Sigma\alpha_i=0$ and

$$|f(x)-\Sigma\alpha_i f_i(x)|<\varepsilon.$$

Hence $|f(x)|<\Sigma\alpha_i|f_i(x)|+\varepsilon$

$$\leq\|\hat{x}\|+\varepsilon.$$

This implies $\|x\|<\|\hat{x}\|+\varepsilon$.

Since ε is arbitrary, it follows that $\|x\|\leq\|\hat{x}\|$.

Therefore $\|x\|=\|\hat{x}\|$ and θ is isometric.

It is clear that for any element x of K $\hat{x}$ is nonnegative on Ext P. Conversely let x be an element of H such that $\hat{x}$ is nonnegative on Ext P. Then for any positive functional f on H $\hat{x}(f)\geq 0$ since $\|f\|^{-1}f$ lies in P and by the Klein-Milman Theorem $\|f\|^{-1}f$ is contained in the weak closure of the convex extension of the set of extremal points of P. It can be proved——see, *e.g.*, (3)——that the set $\{x: f(x)\geq 0$ for all $f\in K^*\}$ is the closure of K in the strongest locally convex topology. Since K is closed in this topology, x lies in K and θ is an order isomorphism.

Q. E. D.

Lemma 1. Let (X, K, $\|x\|$) be an ordered Banach space. Then (X, K, $\|x\|$) is an M space with order unit if and only if (X, K) is a vector lattice and (X, K, $\|x\|$) is a GM space.

Proof. The neccessity follows immediately from Theorm 1.1.

Now assume that (X, K, $\|x\|$) is a GM space with e as the norm-inducing order unit and that (X, K) is a vector lattice. Then $-y\leq x\leq y$ if and only if $|x|\leq|y|$. Hance the GM space norm can be difined as $\|x\|=\inf\{t>0: |x|\leq te\}$ and so $\|x\|=\|y\|$ if $|x|=|y|$. Since $\|x\|$ is monotone, $\|x\|$ is a lattice norm. It can be easily verified that $\|x\vee y\|=\|x\|\vee\|y\|$ for any element x and y in K. This completes the proof of sufficiency.

Q. E. D.

Lemma 2. Let (X, K, $\|x\|$) be an M space with order unit e and f be a nonzero functional in P. Then following conditions are equivalent.

1) f is a lattice functional and $f(e)=1$.

2) For any element g in P such that $0\leq g\leq f$ it is ture that $g=g(e)f$.

(3) f is an element of Ext P.

Proof. This is the direct application of Lemma 1 and (2; 24-1 and 24-2).

Lemma 3. Let A be a complex B* algebra with identity e such that (H, k) is a vector lattice. Let A_1 be any closed subalgebra containing e. Then for any f in Ext P the restriction f_0 of f on H_1 is an element of Ext P_1.

Proof. Notice that (H, K) and (H_1, K_1) are M spaces with e as order unit under the B* norm.

Since (H, K) is a vector lattice, its order dual (K^*-K^*, K^*) is a vector lattice. Let g_0 be an element in P_1 such that $0\leq g_0\leq f_0$. There exists an element g' in P which is an extension of g_0 —*e. g.*, see (1; 4-7-11). Let $g=g'\wedge f$. Then g is also an extension of g_0 and contained in P. Since $0\leq g\leq f$ and f is in Ext P, $g=g(e)f$ by Lemma 2. Hence $g_0=g_0(e)f_0$ and it follows that f_0 is an element of Ext P_1.

Q. E. D.

Lemma 4. Let A be a comhlex B* algebra with identity e such that (H, K) is a vector lattice. Then A is commutative.

Proof. Let h be any element in H and A_1 be the closed subalgebra generated by h and e. Then A_1 is commutative and *-isomorphic to C(Ext P_1) by Theorem 2. 1. Hence by the previous Lemma 3 for any f in Ext P it is ture that $f(h^2)=(f(h))^2$.

Let f be any element in Ext P. Remember that $f(x)=\overline{f(x^*)}$. Let x be an element of A such that $f(x)=0$. x can be written $x=h+ik$ where h and k are in H. Since $f(x)=0$ and $f(h)$ and $f(k)$ are real, $f(h)=f(k)=0$. Hence $f(h^2)=f(k^2)=0$. On the while since $f(h+k)=0$,

$$f((h+k)^2)=f(hk+kh)=0.$$

This shows that $f(hk)=\overline{f(kh)}=\beta i$ where β is a real number. Suppose that $\beta\neq 0$. Then

$$f(x^*x)=if(hk-kh)=i(f(hk)+f(hk))=-2\beta.$$

Since $f(x^*x)\geq 0$, β must be negative. However, if we set $y=k+ih$, then $f(y^*y)=2\beta<0$ which is a contradiction. Hence $\beta=0$ and $f(x^*x)=0$. Since $|f(x)|^2\leq f(e)f(x^*x)$ for any x in A, it follows that $f(x^*x)=0$ if and only if $f(x)=0$ for any f in Ext P.

Now for any f in Ext P let $L(f)=\{x : f(x^*x)=0\}$. Let $x \to T_x$ be the *-representation associated with f on the Hilbert space A-L(f) such that $f(x)=(T_x e', e')$ where e' is $e+L(f)$ (cf. 1; 4-3-7). By the previous discussion $L(f)$ is the kernel of f and hence A-L(f) is one dimensional. For any x and y in A let $T_x e'=\lambda e'$ and $T_y e'=\mu e'$. Then

$$T_{xy}e'=\lambda\mu e'=T_{yx}e'$$

and it follows that $f(xy)=f(yx)$.

Above discussions show that the isometric order isomorphism of A to C(Ext P) given as in Theorem 2.1 is an algebraic isomorphism from which the commutativity of A directly follows.

Q. E. D.

Remark. In Lemma 4, the assumption that (H, K) bea vector lattice is a little superfluous. In fact, equivalence of 2) and 3) in Lemma 2 holds for any GM space. Lemma 3 is true if (H, K) has the decomposition property only. Hence Lemma 4 holds ture if (H, K) has the decomposition property. All the proofs are unaltered in these cases.

Theorem 2. 2. Let A be a complex *-algebra with identity e. Then A is a commutatative B* algebra under a suitable norm if and only if H has a norm making (H, K) into an M space with e as order unit. In this case the M space norm coincides with the B* norm restricted on H.

Proof. Let A be commutative B* algebra with identity e. By Theorem 2.1 A is isometric *-isomorphic onto C(Ext P) and hence (H, K) is order-isomorphic onto C*(Ext P) under the evaluation map $x \to \hat{x}$. Since $f(e)=1$ for any f in Ext P, C*(Ext P) is clearly an M space with e as order unit. Therefore the B* norm restricted on H makes (H, K) into an M space with e acting as order unit.

Conversely let $\|x\|$ be the norm making (H, K) into an M space with e acting as order unit. By the previous Lemma 1 (H, K, $\|x\|$) is a GM space and (H, K) is a vector lattice. For any element y of A define a norm $\|y\|_1$ such that $\|y\|_1^2=\|y^*y\|$. They by Theorem 1.2 the norm $\|y\|_1$ is a complete B* norm for A which coincides with the norm $\|y\|$ on H. Moreover, since (H, K) is a vector lattice, by Lemma 4 A must be commutative.

Q. E. D.

Theorem 2. 3. Let $A_i (i=1, 2)$ be commutative complex B* algebras with identities e_i and (H_i, K_i) be the corresponding ordered linear spaces of hermitian elements of A_i. Let θ be an order isomorphism of (H_1, K_1) onto (H_2, K_2) taking e_1 onto e_2. Then the linear extention of θ to

$$\bar{\theta} : A_1 \to A_2$$

is the isometric *-isomorphism of A_1 onto A_2.

Proof. Since a B* algebra has a unique B* norm, isometry follows immediately once *-isomorphism is proved.

Remark that since H_i are subalgebras of A_i, the proof will be completed if we show that θ is an algebraic homomorphism.

Let H_i' be the dual spaces of H_i and $\theta' : H_2' \to H_1'$ be the dual map of θ defined as $\theta' f_2(x)=f_2(\theta x)$ for all x in H_1 and f_2 in H_2'. Then since θ is an order isomorphism of H_1 on H_2 taking e_1 onto e_2, it follows easily that $\theta'(P_2)=P_1$ and $\theta'(\text{Ext } P_2)=\text{Ext } P_1$.

Now by Theorem 2.1. (H_i, K_i) are algebraically isomorphic onto C*(Ext P_i) under the evaluation map $x_i \to \hat{x}_i$. If x_1 and y_1 are arbitrary elements of H_1, then for any element f_2 in Ext P_2

$$\begin{aligned}\widehat{\theta(x_1y_1)}(f_2) &= f_2(\theta(x_1y_1))=\theta' f_2(x_1y_1)\\ &= \widehat{x_1y_1}(\theta' f_2)=\hat{x}_1(\theta' f_2)\hat{y}_1(\theta' f_2)\\ &= \theta' f_2(x_1)\theta' f_2(y_1)=f_2(\theta x_1)f_2(\theta y_1)\\ &= \widehat{\theta x_1}(f_2)\widehat{\theta y_1}(f_2)=(\widehat{\theta x_1}\ \widehat{\theta y_1})(f_2)\\ &= \widehat{(\theta x_1)(\theta y_1)}(f_2)\end{aligned}$$

Hence $\theta(x_1y_1)=(\theta x_1)(\theta x_2)$ and θ is an algebraic homomorphism.

Q. E. D.

Remark. The previous proof is based on the fact that the carrier space of a commutative complex B* algebra with identity is completely determined by the order and linear structure of the algebra.

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