◇論　文◇

ON SINGULAR MATRICES

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1. Penrose [7] discussed a generalized inverse for matrices, and he established the following theorem.

Theorem A. For any matrix A, the four equations AXA=A, XAX=X, $(AX)^*=AX$, and $(XA)^*=XA$ have a unique solution X, where A^* denotes the conjugate transpose of A.

This unique solution X is called the generalized inverse of A. If we remove the third and fourth equations in Theorem A above, a solution X (of the equations XAX=X and AXA=A) is not in general unique.

Then the natural question is:

Problem. What is the cardinal number of the set of all solutions X of the equations AXA=A and XAX=X for a matrix A in the set $M_n(F)$ of all n by n matrices over a field F?

The purpose of this note is to prove (Theorem 1) that if $A\in M_n(F)$ then the cardinal number of the set of all solutions X of the equations AXA=A and XAX=X is equal to $|F|^{2(\text{rank of } A)(n-(\text{rank of } A))}$

This result gives us a new definition of a regular semigroup (see Definition 2) and new regular semigroups with zero (see Theorem 2).

2. Let F be a field. $M_n(F)$ denotes the set of all n by n matrices over the filed F with binary operation, the usual metrix multiplication. By Theorem A, $M_n(F)$ is a regular semigroup. We define $V(A)=\{X\in M_n(F): AXA=A \text{ and } XAX=X\}$ which will be called an inverse set of A in $M_n(F)$. $\rho(A)$ denotes the rank of a matrix A in $M_n(F)$, and $|T|$ denotes the cardinal number of a set T.

Lemma 1. Let $A\in M_n(F)$ and let $X\in V(A)$. Then $\rho(A)=\rho(X)$.

Proof. Form AXA=A and XAX=X, $\rho(A)=\rho(AXA)\leq\rho(X)=\rho(XAX)\leq\rho(X)$ by Theorem 1.4 of [6, p.83]; hence $\rho(A)=\rho(X)$.

Lemma 2. The cardinal number of an inverse set V(A) of a matrix A in $M_n(F)$ is invarant under elementary row or column operation on A, that is, $|V(A)|=|V(EA)|=|V(AH)|$, where E and H are elementary matrices (see Definition of elementary matrices on page 91 in [6]).

Proof. Let $A\in M_n(F)$ and let E be an elementary matrix in $M_n(F)$. Let $X\in V(A)$ and let E^{-1} be the inverse matrix of the non-singular matrix E. Then $EA=E(AXA)=EA(XA^{-1})EA$ and $XE^{-1}=(XAX)E^{-1}=XE^{-1}(EA)XE^{-1}$; hence $V(A)E^{-1}\subset V(EA)$ and $|V(A)|\leq|V(EA)|$. Similarly, we obtain $V(EA)E\subset V(A)$ and $|V(EA)|\leq|V(A)|$. Thus $|V(A)|=|V(EA)|$. Analogously, we have $|V(A)|=|V(AH)|$, where H is an elementary matrix. This proves Lemma 2.

We need the following well known theorem.

Theorem B. Every m by n matrix A is equivalent to a matrix $C=(c_{ij})$ where $c_{ii}=1$, $i=1, 2, \dots, \rho(A)$, and $c_{ij}=0$, otherwise. The matrix C is called the canonical form of A (see Theorem 3.4 on page 106 in [6]).

For $1\leq k\leq n$. Let $C_k=(d_{ij})$ where $d_{ii}=1$ for $i=1, 2, \dots, k$ and $d_{ij}=0$, otherwise.

According to Lemma 2 and Theorem B, to solve the problem we need only consider c_k, $k=1, 2, \dots n$.

The main lemma follows.

Lemma 3. Let k and n be positive integers with $k\leq n$. Let F_q be a Galois field with q elements. If $C_k\in M_n(F_q)$, then $|V(C_k)|=q^{2k(n-k)}=q^{2(\rho(C_k)(n-\rho(C_k)))}$

Proof. Let $k<n$. Let X be an element of the inverse set $V(C_k)$. Then $C_kXC_k=C_k$ and $XC_kX=X$.

By direct calculation, it is not hard to see that $X=(x_{ij})$ takes the form:

$$x_{ij}=\begin{cases}1 \text{ if } i=j \text{ and } i=1,2,\dots,k;\\ 0 \text{ if } i\neq j \text{ and } \{i,j\}\subset\{1,2,\dots,k\};\\ x_{ij} \text{ if } i=1,2,\dots,k \text{ and } j=k+1,k+2,\dots,n;\\ x_{ij} \text{ if } i=k+1,k+2,\dots,n \text{ and } j=1,2,\dots,k;\\ \sum_{t=1}^{k} x_{it}x_{tj} \text{ if } \{i,j\}\subset\{k+1,k+2,\dots,n\},\end{cases}$$

where x_{kj} above are arbitrary in F_q. Thus we are able to choose $2k(n-k)$ entries of X arbitrary so that the cardinal number of the set $V(C_k)$ is equal to $q^{2k(n-k)}$. If $k=n$, then $V(C_n)=\{C_n\}$, and $|V(C_n)|=1$. This proves Lemma 3.

Theorm 1. If $A\in M_n(F)$, then the cardinal number of the inverse set $V(A)$ is equal to $|F|^{2\rho(A)(n-\rho(A))}$.

Proof follows from Lemmas 2, 3 and Theorem B.

3. Applitications and a question.

Definition 1. A semigroup S with 0 is said to be *homogeneous n regular* if $|V(a)|=n$ for every $a\subset S\backslash 0$ [4].

Let n and k be two positive integers with $k\leqq n$. We define $S_{n,k}(F)=\{X\in M_n(F): \rho(X)\leqq k\}$, and let $S_{n,n-1}(F)=S_n(F)$.

We have corollaries and Theorem 2.

Corollay 1. $S_2(F_q)$ is a homogeneous q^2 regular semigroup with 0, where F_q is a finfte field with q elements.

$S_3(F_q)$ is a homogeneous q^4 regular semigroup with 0.

$S_{n,1}(F_q)$ is a homogeneous $q^{2(n-1)}$ regular semigroup with 0.

Corollary 2. If F is a field of charcteristic 0 then $S_n(F)$ is a homogeneous ∞ regular semigroup with 0.

We have a new definition of a regular semigroup with 0.

Definition 2. Let S be a regular semigroup with 0. S is called a $[s, t]$ *regular semigroup with* 0 if $s\leqq|V(a)|\leqq t$ for every $a\in S\backslash 0$, where s and t are positive integers with $s<t$.

Theorem 2. Lef F_q be a Galois field with q elements. Then $S_n(F_q)$ is a $[q^{2(n-1)}, q^{2[n/2](n-[n/2])}]$ regular semigroup with 0, where

$$[n/2]=\begin{cases}n\text{-}2 & \text{if } n \text{ is even},\\ n\text{-}1/2 & \text{if } n \text{ is odd}.\end{cases}$$

In $S_3(F_q)$, there are two non-zero idempotents $e=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ and $f=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ with $ef=fe=e$ and $e\neq f$. Hence f is not a primitive idempotent of the homogeneous q^4 regular semigroup $S_3(F_q)$. This example shows that the condition "every idempotent of S is primitive" is not necessary for a regular semigroup S with to be homogeneous n regular (see Theorems 1, 3, 7 and 8 in [4]).

Hence we raise the following question:

Question. What are necessary and sufficient conditions for a regular semigroup S with 0 be homogeneous n regular?

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References

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys, No. 7, Amer. Math. Soc., Providence, R. I., 1961.

2. R. E. Cline, *Representations for the generalized inverse of a partitioned matrix*, J. Soc. Indust. Appl. Math., vol. 12, no. 3 (1964), pp. 588–600.

3. Henry O. Decell, Jr., *An alternative form of the generalized inverse of an arbitrary complex matrix*, Siam Review, vol. 7 (1965), pp. 356–357.

4. Jin Bai Kim, *Completely 0-simple and homogeneous n regular semigroups*, Bull. Amer. Math. Soc., vol. 71 (1965), pp. 867–871.

5. Jin Bai kim, *Completely 0-simple and homogeneous n regular semigroups*, Proccdings of the Amer. Math. Soc., (to appear), April issue of 1966.

6. Marvin Marcus and Henry Minc, *Introduction to Linear Algebra*, The Macmillan Company, New York, 1965.

7. R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Phil. Soc., 51 (1955), pp. 406–413.

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