CONHARMONIC TRANSFORMATIONS IN RECURRENT AND RICCI-RECURRENT RIEMANNIAN SPACES

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§ 1. Introduction.

Let V_n be the n-dimensional Riemannian space, and R_{ijkl} be the curvature tensor. V_n is recurrent if

(1. 1) $$R_{ijkl,r}=v_r R_{ijkl}$$

where the index of the comma indicates the covariant derivative with respect to the metric g_{ij} and v_r is a gradient.

However if it satisfies

(1. 2) $$R_{ij,r}=v_r R_{ij}, \qquad (R_{ij}=R^k{}_{ijk}).$$

V_n is a Ricci-recurrent. In this case $v_r=0$, i.e. $R_{ijkl,r}=0$ respectively $R_{ij,r}=0$, V_n is a symmetric and Ricci-symmetric Riemannian space respectively.
Contracting (1.2) by g^{ij} we see that for a recurrent and Ricci-recurrent Riemannian space the relation

(1. 3) $$R_{,r}=v_r R$$

is satisfied.

Mileva Prvanovitch [3] have studied projective and conformal transformations in recurrent and Ricci-recurrent Riemannian spaces. Y. Ishii [1] introduces the conharmonic transformation in a Riemannian space, and the auther have studied relations between subgroups of a conformal transformation group.

The present auther wishes to study the conharmonic transformations in recurrent and Ricci-recurrent Riemannian spaces.

§ 2. Conharmonic trantformation.

We consider a conformal transformation

(2. 1) $$\bar{g}_{ij}=e^{2\sigma}g_{ij}$$

of the fundamental tensor g_{ij}, were σ is a scalar function. It is well known that, for a conformal transformation, we have

(2 .2) $$\overline{\{^h_{ij}\}}=\{^h_{ij}\}+\sigma_{,j}\delta^h_j+\sigma_{,j}\delta^h_i-\sigma_{,}{}^h g_{ij}$$

and

(2. 3) $$\bar{R}^h{}_{ijk}=R^h{}_{ijk}+g_{ij}\sigma^h_k-g_{ik}\sigma^h_j+\sigma_{ij}\delta^h_k-\sigma_{ik}\delta^h_j,$$

where

(2. 4) $$\sigma_{ij}=\sigma_{,ij}-\sigma_{,i}\,\sigma_{,j}+\frac{1}{2}\sigma_{,}{}^a\sigma_{,a}g_{ij}.$$

Let A be a harmonic function which is defined by

(2. 5) $$g^{ij}A_{,ij}=0$$

in an n-dimensional ($n>2$) Riemannian space V_n, where $,i$ denote covariant differentiation with respect to the metric tensor $g_{ij,}$ A harmonic function does not in general transform into a harmonic function by a conformal transformation. A conformal transformation is said to be conharmonic one if there exist a conformal transformation such as a harmonic function is transformed into a harmonic function. Evidently these transformations form a subgroup of the conformal transformation group. As is weel-known, the conformal transformation (2.1) to be conharmonic is that there exist a scalar function such that

(2. 6) $$\sigma^p_p=g^{ij}\sigma_{ij}=\sigma^p_{.p}+\frac{1}{2}(n-2)\sigma_{,i}\sigma_{,}{}^i=0\,.$$

Contracting (2.3) with respect to the indices h and k, we have

(2. 7) $$\bar{R}_{ij}=R_{ij}+(n-2)\sigma_{ij}.$$

By means of

(2. 8) $$\bar{g}^{ij}=e^{-2\sigma}g^{ij},$$

the scalar curvature is given by

(2. 9) $$\bar{R}=e^{-2\sigma}R.$$

From (2.1) and (2.9), we obtain an invariant relation

(2.10) $$\bar{g}_{ij}\bar{R}=g_{ij}R.$$

We have seen that (2.10) is the necessary and sufficient condition in order that the conformal transformation (2.1) be conharmonic.

From (2.6), we find

(2. 11) $$g^{ij}\sigma_{,ij}=\sigma^p_{,p}=\frac{2-n}{2}\sigma_{,p}\sigma_{,}{}^p\leqq 0,\ \ (n\geqq 2).$$

In a compact Riemannian space with positive definite metric, making use of the theorem of Hopf-Bochner [6] we have

(2. 12) $\sigma=\text{const}.$

Thus we have

THEOREM 1. *In a compact Riemannian space with positive definite metric, the conharmonic transformation is a homothetic.*

Next, we shall deduce some identities which are useful in the later sections.

Solving (2.7) for σ_{ij} and analogous expressions σ_{ik}, σ^h_k and σ^h_j being substituted to (2.3), then we obtain

$$\bar{C}^h{}_{ijk}=C^h{}_{ijk}, \tag{2.13}$$

where

$$C^h{}_{ijk}=R^h{}_{ijk}+\frac{1}{n-2}(\delta^h_j R_{ik}-\delta^h_k R_{ij}+g_{ik}R^h_j-g_{ij}R^h_k), \tag{2.14}$$

which is invariant under a conharmonic transformation. Contracting (2.14) with respect to the indices h and k, we obtain

$$C_{ij}=\frac{1}{2-n}\ g_{ij}R, \tag{2.15}$$

which is also invariant under a conharmonic transformation.

The conformal curvature tensor of Weyl may be expressed by

$$W^h{}_{ijk}=C^h{}_{ijk}-\frac{1}{n-1}(\delta^h_k C_{ij}-\delta^h_j C_{ik}), \tag{2.16}$$

§ 3. Recurrent Riemannian spaces.

In this section, we shall obtain a necessary and sufficient codition that a recurrent Riemannian space be transformed into a recurrent Riemannian space by the conharmonic transformation.

If the spaces V_n and $\bar{V}_n$ are recurrent Riemannian spaces, then from the definition, we get

$$C^h{}_{ijk,l}=v_l C^h{}_{ijk}, \tag{3. 1}$$

$$\bar{C}^h{}_{ijk/l}=\bar{v}_l \bar{C}^h{}_{ijk}, \tag{3. 2}$$

where $/l$ denotes covariant differentiation with respect to the metric tensor $\bar{g}_{ij}$ in $\bar{V}_n$.

Differentiating (2.11) covariantly and taking account of (3.1) and (3.2), we have

(3. 3) $$\bar{C}^h{}_{ijk}\bar{v}_l = C^h{}_{ijk}v_l - 2C^h{}_{ijk}\sigma_l + C^a{}_{ijk}\sigma_a\delta^h_l - C^a{}_{ijk}\sigma^h g_{al} - C^h{}_{ljk}\sigma_i$$

$$-C^h{}_{ilk}\sigma_j - C^h{}_{ijl}\sigma_k + (C^h{}_{ajk}g_{il} + C^h{}_{iak}g_{jl} + C^h{}_{ija}g_{kl})\sigma^a,$$

On the other hand, dieferentiating (2.9) covariantly, we get

(3. 4) $$\bar{R}_{/l} = e^{-2\sigma}(R_{,l} - 2R\sigma_l).$$

In a recurrent Riemannian space, (1.3) hold good, hence we get

(3. 5) $$\bar{v}_l = v_l - 2\sigma_l.$$

Thus, (3.3) can be written

(3. 6) $$C^a{}_{ijk}\sigma_a\delta^h_l - C^a{}_{ijk}\sigma^h g_{al} - C^h{}_{ljk}\sigma_i - C^h{}_{ilk}\sigma_j - C^h{}_{ijl}\sigma_k$$

$$+ (C^h{}_{ajk}g_{il} + C^h{}_{iak}g_{jl} + C^h{}_{ija}g_{kl})\sigma^a = 0.$$

Contracting for h and l, we find

(3. 7) $$(n-1)C_{iajk}\sigma_a - (C_{aijk} + C_{jiak} + C_{kija})\sigma^a + C_{ik}\sigma_j - C_{ij}\sigma_k = 0,$$

where

(3. 8) $$C_{iajk} = g_{ih}C^h{}_{ajk}.$$

From (2.14) and (3.8), we find

(3. 9) $$C_{iajk} + C_{jiak} + C_{kija} = C_{iajk} + C_{ijka} + C_{ikaj} = 0.$$

Thus, (3.7) can be written

(3.10) $$C^a{}_{ijk}\sigma_a = \frac{1}{n-1}(C_{ij}\sigma_k - C_{ik}\sigma_j).$$

Transvecting this equation with σ^i and taking account of (2.15), we get

(3.11) $$C^a{}_{ijk}\sigma_a\sigma^i = 0.$$

Multiplying (3.6) by σ_h and using of (3.10) and (3.11), we get

(3.12) $$[C^h{}_{ijk} - \frac{1}{n-1}(C_{ij}\delta^h_k - C_{ik}\delta^h_j)]\sigma_a\sigma^a = 0,$$

which is equivalent to

(3.13) $$W^h{}_{ijk}\sigma_a\sigma^a = 0,$$

and this means that either $\sigma_a = 0$, i.e. the transformatiom under consideration is homothetic, or $W^h{}_{ijk} = 0$ for $n > 2$.

Conversely, if $\sigma_a = 0$, then from (2.2) and (2.3), we have

(3.14) $$\overline{\{^{h}_{ij}\}} = \{^{h}_{ij}\},$$

(3.15) $$\bar{R}^{h}{}_{ijk}=R^{h}{}_{ijk}.$$

Differentiating (3.15) covariantly and taking account of (3.14) and (3.15), we get

(3.16) $$\bar{R}^{h}{}_{ijk/l}=R^{h}{}_{ijk,l}.$$

If V_n is a recurrent Riemannian space, then from (1.1), we get

(3.17) $$\bar{R}^{h}{}_{ijk/l}=R^{h}{}_{ijk,l}=R^{h}{}_{ijk}v_l=\bar{R}^{h}{}_{ijk}v_l,$$

which shows that $\bar{V}_n$ is a also recurrent Riemannian space.

Next, we suppose that the space V_n be a conformal Euclidean, i.e. $W^{h}{}_{ijk}=0$. Since the conformal curvature tensor is invariant under a conformal transformation, we find that $\bar{W}^{h}{}_{ijk}=0$.

Thus we have

THEOREM 2. *A necessary and sufficient condition that a recurrent Riemannian space be transformed into a recurrent Riemannian space by a conharmonic transformation is that either the space is a conformal Euclidean or the transformation is a homothetic.*

Next, in a compact Riemannian space, the homothetic transformation is always an isometric. Thus we have

THEOREM 3. *A necessay and sufficient condition that the compact recurrent Riemannian space be transformed into a compact recurrent one by a conharmonic transformation is that the transformation is an isometic.*

Now, differentiating (3.5) covariantly, we get

$$\bar{v}_{l/m}=v_{l,m}-2\sigma_{l,m}-v_m\sigma_l-v_l\sigma_m+g_{lm}\sigma^a v_a+2(2\sigma_l\sigma_m-g_{lm}\sigma_a\sigma^a).$$

Transvecting this equation with $\bar{g}^{lm}$ and using of (2.6), we get

(3.18) $$\bar{v}^{p}_{p}=e^{-2\sigma}[v^{p}_{,p}+(n-2)(v_p\sigma^p-\sigma_p\sigma^p)].$$

From (3.5), we obtain

(3.19) $$\bar{v}_p\bar{v}^p=e^{-2\sigma}(v_pv^p-4v_p\sigma^p+4\sigma_p\sigma^p).$$

In a recurrent Riemannian space, if we put

(3.20) $$v^{p}{}_{p}=v^{p}{}_{,p}+\frac{n-2}{4}v_pv^p,$$

then, from (3.18) and (3.19), we find

(3.21) $$\bar{v}^p{}_p = e^{-2\sigma} v^p{}_p.$$

Transvecting this equation with g_{ij}. we have

(3.22) $$\bar{g}_{ij}\bar{v}^p{}_p = g_{ij}v^p{}_p.$$

Thus we have

THEOREM 4. *In a recurrent Riemannian space,* $g_{ij}v^p{}_p$ *is invariant under a conharmonic transformation.*

§ 4. Ricci-recurrent Riemannian spaces.

Let V_n be a Ricci-recurrent Riemannian space, that is,

(4. 1) $$R_{ij,l} = v_l R_{ij},$$

where R_{ij} is a Ricci tensor and v_l is a gradiant.

We suppose that V_n be transformed into $\bar{V}_n$ by a conharmonic transformation and $\bar{V}$ be also a Ricci recurrent Riemannian space, then

(4. 2) $$\bar{R}_{ij/l} = v_l \bar{R}_{ij}.$$

Under a conharmonic transformation, The Ricci tensor R_{ij} will be transformed into

(4. 3) $$\bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij}.$$

Differentiating (4.3) covariantly and using of (4.1), (4.2) and (3.6), we have

(4. 4) $$(n-2)(\sigma_{ij,l} - \sigma_{ij}v_l) - (R_{lj}\sigma_i + R_{il}\sigma_j - R_{ai}\sigma^a g_{jl} - R_{ja}\sigma^a g_{il})$$
$$= (n-2)(\sigma_{lj}\sigma_i + \sigma_{li}\sigma_j - \sigma_{ai}\sigma^a g_{jl} - \sigma_{ja}\sigma^a g_{il}).$$

Multiplying (4.4) by σ^l and summing over l, we get

(4. 5) $$(\sigma_{ij,l} - \sigma_{ij}v_l)\sigma^l = 0.$$

Thus we have

THEOREM 5. *In a Ricci-recurrent Riemannian space, The conharmonic transformation satisfies* (4.5)

Next, we suppose that V_n be a Ricci-symmetric Riemannian space, that is,

(4. 6) $$R_{ij,l}=0,$$

and, from (3.6), we have $\sigma_l=0$. Hence we have

THEOREM 6. *A necessary and sufficient condition that a Ricci-symmetric space be transformed into a Ricci-symmetric space is that the conhamonic transformation is a homothetic.*

By the theorem of Hopf-Bochner, the homothetic transformation in a compact Riemannian space is always an isometric [5]. Thus we have

THEOREM 7. *A necessary and sufficient condition that a compact Ricci symmetric space be transformed into a compact Ricci symmetric space is that the conharmonic transformation is an isometric.*

§ 5. **Einstein spaces.**

We suppose that V_n and $\bar{V}_n$ be Einstien spaces with non-zero curvature scalar, that is,

(5. 1) $$R_{ij}=\frac{R}{n}g_{ij},$$

(5. 2) $$\bar{R}_{ij}=\frac{\bar{R}}{n}\bar{g}_{ij}.$$

From (2.10), we have

(5. 3) $$\bar{R}_{ij}=R_{ij},$$

which shows that the Ricci tensor is preserved by a conharmonic. From (2.7), we get

(5. 4) $$\sigma_{ij}=0 \qquad (n>2).$$

Thus we have

THEOREM 8. *In a Einstein space with non-zero curvature scalar, the conharmonic transformation satisfies* (5.4).

Next, if, in the Einstein space with non-zero constant curvature, then we have

(5. 5) $$R_{ij,l}=0,$$

which shows that an Einstein space with non-zero constant curvature is a Ricci symmetric Riemannian space.

Let V_n be an also Einstein space with non-zero constant curvature, then we obtain

(5. 6) $\bar{R}_{ij/l}=0.$

It follows from (5.5) and (5.6) that $v_l=\bar{v}_l=0$. By virture of (3.5), we get $\sigma_l=0$, i.e. a conharmonic transformation is a homothetic. If, in an Einstein space with non-zero constant curvature, the transformation is a homothetic, then a homothetic transformation is an isometric [5]. Thus we have

THEOREM 9. *In an Einstein space with non-zero constant curvature, a conharmonic transformation is an isometric.*

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