DIRECT DECOMPOSITIONS OF LATTICES OF CONTINUOUS FUNCTIONS

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Introduction. If X is a topological space and if K is a chain equipped with its oder topology, then we denote by $C(X, K)$ the lattice of all continuous functions from X to K. In the paper [I] R.L. Blair and C.W. Burill proved following theorem.

THEOREM B. *Let X be a topological space, let K be a chain equipped with its oder topology, and let L be an adequate sublattices of $C(X, K)$. If L is isomorphic to the direct product of two lattices L_1 and L_2, then X is the union of disjoint open and closed sets X_1 and X_2 having the property that L_i is isomorphic to L_{X_i} $(i=1, 2)$. Moreover, X_i is non empty if and only if L_i has only two distinct elements.*

And in the paper they defined an adequate sublattice as following.

DEFINITION. *A sublattice L of $C(X, K)$ will be called adequate in case for each $x \in X$ there exist functions $f, g \in L$ such that $f(x) > g(x)$.*

In the paper Page. 634 they stated following problem.

PROBLEM. *What are necessary and sufficiant conditions on a sublattice L of $C(X, K)$ in oder that a direct decomposition of L be reflected in a corresponding decomposition of X? In any case, the hypothesis of adequacy cannot simply be deleted.*

So in this paper I modify condition adequate as sub-adequate (defined in this paper) and prove THEOREM I, which is a little expenion of THEOREM B. Also, if X is a topological space such that for each $x \in X$ their exists the least open and closed set containing it, we prove THEOREM 2, in which we use hypothesis of r-adequacy which is weeker than sub-adequacy. The technic used for the proof of THEOREM I, is similar with the technic which is used in the proof of THEOREM B, hence it can be obtained easily if we use THEOREM B, but for the convenience of reader I prove it independently.

Theorems.

For the proof of therems, we include here some properties of lattice which can be found in [1] or [2]. By a prime ideal of a lattice L we mean a non empty proper subset P of L such that (i) if $a, b \in P$ then $a \cup b \in P$ and (ii) $a \cap b \subset P$ if and only if $a \in P$ or $b \in P$, a dual prime ideal is the complement of a prime ideal. If L_1 and L_2 are lattices and P is a prime ideal of the direct product $L_1 \times L_2$, then either $P=P_1 \times L_2$ for some prime ideal P_1 of L_1 or $P=L_1 \times P_2$ for some prime ideal P_2 of L_2. If Y is a subset of X and if $f \in C(X, K)$, then $f \mid Y$ denotes the restriction of f to Y. If L is a sublattice of $C(X, K)$ then we set $L_Y=\{f \mid Y : f \in L\}$. It is clear that L_Y is a sublattice of $C(Y, K)$. We define the terminology which is used for the proof of theorems.

DEFINITION. A non void subset M of X will be called as a *cutting set of X* if and only if there exist two subsets X_1, X_2 of X such that $X_1 \cap X_2=\phi$ (void), $\bar{X}_1 \cup \bar{X}_2=X$, and $\bar{X}_1 \cap \bar{X}_2=M$. ($\bar{X}_i$ denotes the closure of X_i).

DEFINITION. If L is a sublattice of $C(X, K)$ with following two conditions, then we call L a *sub-adequate sublattice.*

(1) For a dense subset D of X and for each $x \in D$ there exist functions $f, g \in L$ such that $f(x) > g(x)$.

(2) Let M be a cutting set of X such that $X_1 \cap X_2=\phi$, $\bar{X}_1 \cup \bar{X}_2=X$ and $\bar{X}_1 \cap \bar{X}_2=M$. If $f(x)=g(x)$ for each pair $(f, g) \in L$, for each $x \in M$ then there exist two functions $f_1 \in L_{\bar{X}_1}$ and $g_2 \in L_{\bar{X}_2}$ such that the extension of (f_1, g_2) on X is not contained in L. (Extension of (f_1, g_2) means the function $h(x) =f_1(x)$ if $x \in \bar{X}_1$ and $h(x)=g_2(x)$ if $x \in \bar{X}_2$).

THEOREM I. *Let X be a topological space, let K be a chain equipped with its order topology, and let L be a sub-adequate sublattice of* $C(X, K)$. If L is *isomorphic to the direct product of two lattices* L_1 *and* L_2, *then X is the union of open and closed subsets* X_1 *and* X_2 *having the property that* L_i *is isomorphic to* L_{X_i} $(i=1,2)$. *Moreover,* X_i *is non empty if and only if* L_i *has two distinct elements.*

PROOF. Since L is sub-adequate, there is a dense subset D such that $x \in D$

if and only if $f(x)>g(x)$, for some pair $(f, g) \in L$. If $x \in D$ and $f \in L$, we set

$$P_x(f)=\{g \in L : g(x) \leqq f(x)\}$$

and

$$P^x(f)=\{g \in L : g(x) \geqq f(x)\}$$

It is clear that $P_x(f)$ (resp $P^x(f)$) is a prime ideal of L provided that it is a proper subset of L. Since $x \in D$, we have either $P_x(f)$ is a prime ideal of L or $P^x(f)$ is a dual prime ideal of L. We choose an isomorphism δ from L onto $L_1 \times L_2$ and fix an element $k \in L$. Denote by $\mathfrak{P}_1$ (resp $\mathfrak{P}_2$) the collection of all prime ideals of L such that $\delta(P)$ is of the form $P_1 \times L_2$ (resp $L_1 \times P_2$) with P_i a prime ideal of L_i. For $i=1,2$, denote by D_i the set of all $x \in D$ such that either $P_x(k) \in \mathfrak{P}_i$ or $L-P^x(k) \in \mathfrak{P}_i$. Then it is easily seen that D_1 and D_2 are disjoint and that $D=D_1 \cup D_2$. We shall show that $\bar{D}_1 \cap D_2 = D_1 \cap \bar{D}_2 = \phi$. For $\bar{D}_1 \cap D_2 = \phi$, if $y \in \bar{D}_1$ and $y \notin D$ then clearly $y \notin D_2$. In the case $y \in \bar{D}_1$ and $y \in D$, then

$$\cap \{P_x(k) : x \in D_1\} \subseteq P_y(k)$$

and

$$\cap \{P^x(x) : x \in D_1\} \subseteq P^y(k),$$

from this we have $y \notin D_2$.
Hence we have $\bar{D}_1 \cap D_2 = \phi$ and similarly $D_1 \cap \bar{D}_2 = \phi$. We know that (1) D is a dense subset of X, (2) $D=D_1 \cup D_2$, $D_1 \cap D_2 = \phi$, (3) $\bar{D}_1 \cap D_2 = D_1 \cap \bar{D} = \phi$ and (4) $\bar{D}_1 \cup \bar{D}_2 = \bar{D} = X$. Let us suppose $M(M=\bar{D}_1 \cap \bar{D}_2)$ is not a void set, then we have $M \subset X-D$ and M is a cutting set. Hence by hypothesis of sub-adequacy there exist two functions $f, g \in L$ such that $f_1 \in L_{\bar{D}_1}$, $g_2 \in L_{\bar{D}_2}$ and the extension of (f_1, g_2) on X is not contained in L. Now π_i be the projection of $L_1 \times L_2$ onto L_i and consider the mapping $\phi_i = \pi_i \cdot \delta$ from L onto L_i. Let $f, g \in L$ and suppose that $\phi_1(f) \leqq \phi_1(g)$ but $f(x)>g(x)$ for some $x \in D_1$. Then $P_x(g)$ is a prime ideal of L that contain g but not f. If $P_x(k) \neq L$, then, since $P_x(g) \cap P_x(k)$ contains a prime ideal $P_x(g \cap k)$. $P_x(g)$ must map onto $P_1 \times L_2$ for some prime ideal P_1 of L_1. But then $\phi_1(f) \in P_1$ so that $f \in P_x(g)$ a contradiction. Moreover, if $P_x(k) \neq L$, then a dual argument again yields a contradiction. Arguing similarly for D_2, we therefore conclude that

(1) $$\phi_i(f) \leqq \phi_i(g) \text{ implies } f|D_i \leqq g|D_i \quad (i=1, 2)$$

and

(2) $$\phi_i(f) \leqq \phi_i(g) \text{ implies } f|\bar{D}_i \leqq g|\bar{D}_i \quad (i=1, 2).$$

Now suppose that $f|D_1 \leqq g|D_1$ but that $\phi_1(f) \not\leqq \phi_1(g)$. Since L_1 is distributive, Zorn's lemma provides a prime ideal P_1 in L_1 that contains $\phi_1(g)$ but not $\phi_1(f)$. Let P be the prime ideal in L that maps onto $P_1 \times L_2$. Then $g \in P$ and $f \notin P$. Let $h=\delta^{-1}(\phi_1(f), \phi_1(g))$ so that $h \notin P$. Now $\phi_2(h)=\phi_2(g)$ and therefore by (1), $h|D_2=g|D_2$. But then $f \cap h \leqq g$ so that $f \cap h \in P$, a contradiction. Using a similar argument for ϕ_2 we obtain

(3) $$f|D_i \leqq g|D_i \text{ implies } \phi_i(f) \leqq \phi_i(g)\ (i=1,\ 2)$$

and

(4) $$f|\bar{D}_i \leqq g|\bar{D}_i \text{ implies } \phi_i(f) \leqq \phi_i(g)\ (i=1,\ 2).$$

We conclude from (2) or (3) that $\phi_i : f|\bar{D}_i \longrightarrow \phi_i(f)$ is a well defined map from $L\bar{D}_i$ onto L_i. Moreover by (1) or (2) ϕ_i is one to one and ϕ_i^{-1} is also order preserving. Hence ϕ_i is an isomorphism. (**) Now we prove $M=\bar{D}_1 \cap \bar{D}_2$ must be a void set. If M is not a void set, then there exist functions f_1, g_2 such that $f_1 \in L\bar{D}_1$, $g_2 \in L\bar{D}_2$ and the extension of (f_1, g_2) on X is not contained in L. Where we know ϕ_1 is an isomorphism between L_{D_1} and L_1, and ϕ_2 is an isomorphism between L_{D_2} and L_2. Let $\phi_1(f_1)=m_1$ then $m_1 \in L_1$, and let $\phi_2(g_2)=m_2$ then $m_2 \in L_2$. Hence if we set $h=\delta^{-1}(m_1, m_2)$ then $h \in L$. It follows that $h|\bar{D}_1=f_1$ and $h|\bar{D}_2=g_2$, that is the extension of (f_1, g_2) on X is contained in L. Thus we have a contradiction, hence M is a void set. We have proved that $\bar{D}_1 \cap \bar{D}_2=\phi$ and $X=\bar{D}_1 \cup \bar{D}_2$. If $L_{\bar{D}_i}$ has two distinct elements then there exist functions f, $g \in L$ such that $f|D_i \neq g|D_i$, hence there exists a point $x_i \in D_i$ such that $f(x_i) \neq g(x_i)$. Thus we have $\bar{D}_i \neq \phi$. Conversely if $\bar{D}_i$ is not void then D_i is not void, hence there is a point $x \in D_i$ and $f(x_i) > g(x_i)$ for a pair $(f, g) \in L$. Thus we have $L_{\bar{D}_i}$ contains more than two elements. If we set $X_1=\bar{D}_1$ and $X_2=\bar{D}_2$ then we have the decomposition we need. We proved THEOREM 1.

Now we define some terminologies which will be used in the proof of THEOREM 2.

DEFINITION. A topological space X will be called as a *L.O.C.-space* if for each $x \in X$ there exists the least open and closed set containing x, and if N is the least closed and open set containing a point x, then we call N a *l.o.c.-set.*

We can see a discrete space or a topological space which has only finite number of open and closed sets is a *L.O.C.-space.*

DEFINITION. If L is a sublattice of $C(X, K)$ with following two conditions, then we call it a *r-adequate sublattice.*

(1) If N is a l.o.c.-set of X then L contains some functions f, g such that $f(x) > g(x)$ fore some $x \in N$.

(2) Let M be a cutting set of X such that $X_1 \cap X_2 = \phi$, $\bar{X}_1 \cup \bar{X}_2 = X$ and $\bar{X}_1 \cap \bar{X}_2 = M$. If $f(x) = g(x)$ for each pair f, $g \in L$ and for each $x \in M$, and if L_{X_1} and L_{X_2} both have more than two elements, then there exist two functions $f_1 \in L_{X_1}$ and $g_2 \in L_{X_2}$ such that the extension of (f_1, g_2) on X is not contained in L.

THEOREM 2. *Let X be a L.O.C.-space, let K be a chain equipped with its order topology, and let L be a r-adequate sublattice of $C(X, K)$. If L is isomorphic to the direct product of L_1 and L_2 (both have more than two elements), then X is the union of open and closed subsets X_1 and X_2 having the property that L_i is isomorphic to L_{X_i}. Moreover, X_i is non empty.*

PROOF. Let $D = \{x \in X : f(x) > g(x)\}$ for some $\{f . g \in L\}$. Then we know D is not void set, we consider D as a topological space equipped with relative topology. Clearly L_D is a sub-adequate sublattice of $C(X, K)$. We denot eisomorphism by $\cong$. Then $L \cong L_D$, suppose $L = L_1 \times L_2$, then we have $L_D \cong L_1 \times L_2$. By THEOREM I, D is decomposed into D_1, D_2 such that $D = D_1 \cup D_2$, D_i are open and closed in the space D and $L_{D_i} \cong L_i$.

We set

$$\mathfrak{A} = \{N_\alpha : N_\alpha = \text{l.o.c.-set of } X\},$$

$$\mathfrak{A}_1 = \{N_\alpha : N_\alpha \cap D_2 = \phi,\ N_\alpha \in \mathfrak{A}\},$$

and

$$\mathfrak{A}_2 = \{N_\alpha : N_\alpha \cap D_2 \neq \phi,\ N_\alpha \in \mathfrak{A}\}.$$

Let

$$N_1 = \{x : x \in X - D,\ x \in N_\alpha \text{ for some } N_\alpha \in \mathfrak{A}_1\},$$

and

$$N_2 = \{x : x \in X - D, x \in N_\alpha \text{ for some } N_\alpha \in \mathfrak{A}_2\}.$$

Let $X_1 = D_1 \cup N_1$, and $X_2 = N_2 \cup D_2$.

We shall show $X_1 \cup X_2 = X$ and $X_i (i = 1, 2)$ are two disjoint open and closed sets of X. By the construction it is clear that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \phi$. By the definition of D, for each $x \in D$, there exist two functions $f . g \in L$ such that $f(x)$

$>g(x)$. Since f, g are continuous functions, there is an open set v_x of x, such that for each $x_\alpha \in v_x$, $f(x_\alpha)>g(x_\alpha)$, hence we have $v_x \subset D$. Thus D is an open set of X, and hence D_1 and D_2 are open sets of X. Now let $x \in X_1$, if $x \in D_1$ then D_1 is an open neighborhood of x in X, and if $x \in N_1$, then by the construction of N, there is an l.o.c.-set N_x which contains x, and N_x is contained in X_1, and this N_x is an open set of X. Hence we have X_1 is an open set of X. Since D_2 is an open set in X, we have D_2' (complement of D_2 in X) is a closed set. Where by construction $N_1 \subset D_2'$ hence $\overline{N}_1 \cap D_2=\phi$, and since D_1 and D_2 are disjoint closed sets in the relative topology, we have also $\overline{D}_1 \cap D_2=\phi$. ($\overline{D}_1$ denotes the closure of D_1 in X, in this paper if A is a subset of X, then by $\overline{A}$ we denote the closure of A in X). Since X_1 is an open set, we can see X_2 is a closed set, and hence $X_1 \cap \overline{X}_2=\phi$. Let $M=\overline{X}_1 \cap \overline{X}_2$, then

$$\begin{aligned}\overline{X}_1 \cap \overline{X}_2=\overline{X}_1 \cap X_2 &= (\overline{N}_1 \cup \overline{D}_1) \cap (N_2 \cup D_2)\\ &=(\overline{N}_1 \cap N_2) \cup (\overline{N}_1 \cap D_2) \cup (\overline{D}_1 \cap D_2) \cup (\overline{D}_1 \cap N_2)\\ &=(\overline{N}_1 \cap N_2) \cup (\overline{D}_1 \cap N_2), \text{ (other terms are void set see above)}\\ &=M \subset X-D \qquad (X-D \text{ denotes the complement of } D).\end{aligned}$$

Thus if M is not void then it is a cutting set, and for each $x \in M$, for all pairs $(f. g) \in L$, we have $f(x)=g(x)$. Since $x \in \overline{X}_i$ and $x \notin D_i$ implies $f(x)=g(x)$ for all pair $(f. g) \in L$, we have $L_{X_1} \cong L_{D_1} \cong L_1$ and $L_{X_2} \cong L_{D_2} \cong L_2$. If we recall the (**) part of THEOREM 1, we see that M must be a void set. And also in a similar method as THEOREM 1, we have X_i is not empty, since L_i has two elements. We conclude $X_i (i=1, 2)$ are two disjoint open and closed sets. We proved THEOREM 2.

We can see that all of the two characters of which one is used in THEOREM 1, and the other used in THEOREM 2, are not the necessary conditions of the of the PROBLEM needs.

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