ON ADJOINT COMPLEX STRUCTURES IN THE DIFFERENTIABLE MANIFOLD OF DIMENSION $2n+1$

By Jae Koo Ahn

Introduction.

In the present paper, we define a $(2n+1)$-dimensional differentiable manifold which is liken to a manifold constituted by the $2n$-dimensional complex manifold and one-dimensional real manifold adjoining to it, and, we study this manifold as an analogue of the complex manifold.

In § 1, we define an adjoint complex manifold befitting to the complex manifold, introduce a Riemannian metric to this and define more an adjoint Hermitian manifold. In § 2, we calculate the Christoffel symbol and define an adjoint Kaehlerian manifold befitting to the Kaehlerian manifold. And in § 3, we calculate the curvature tensor in the adjoint Kaehlerian manifold. In § 4, we define an almost adjoint complex manifold introduced by three structure tensors, and find the form of the structure tensors when this manifold induces to the adjoint complex manifold, and a condition that this manifold admits an adjoint complex manifold. In § 5, and § 6, we define an almost adjoint Hermitian manifold and an almost adjoint Kaehlerian manifold, and find the conditions that they admit the adjoint Hermitian manifold and an adjoint Kaehlerian manifold respectively.

§ 1. Adjoint complex manifold.

Let us consider a $(2n+1)$-dimensional differentiable manifold X_{2n+1} of class C^∞, which is covered by the coordinate neighborhood system (x^i), where i, j, k is taken by the indices $1, 2, \cdots\cdots, n; \bar{1}, \bar{2}, \cdots\cdots, \bar{n}; \infty$. For arbitrary point P of X_{2n+1}, let P belong to $U \cap U'$, then it holds

$$(1.1) \qquad x^{i'} = x^{i'}(x), \qquad \left| \frac{\partial x^{i'}}{\partial x^i} \right| \neq 0,$$

where U and U' are coordinate neighborhoods represented by x^i and $x^{i'}$ and $x^{i'}(x)$ is real analytical functions of x^i.

Let us put

(1.2) $$z^{\alpha}=x^{\alpha}+ix^{\bar{\alpha}},\quad z^{\bar{\alpha}}=x^{\alpha}-ix^{\bar{\alpha}},\quad z^{\infty}=x^{\infty},$$

where α, β, γ and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are taken by the indices 1, 2, ..., n and $\bar{1}$, $\bar{2}$, ...$\bar{n}$ respectively, then $(x^{\alpha}, x^{\bar{\alpha}}, x^{\infty})$ assigns to $(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty})$, and conversely by the equations

(1.3) $$x^{\alpha}=\frac{z^{\alpha}+z^{\bar{\alpha}}}{2},\quad x^{\bar{\alpha}}=\frac{z^{\alpha}-z^{\bar{\alpha}}}{2i},\quad x^{\infty}=z^{\infty},$$

Form this, we have a one-to-one correspondence between (x^{i}) and $(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty})$, and $(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty})$ may be considered to be a coordinate of point P in X_{2n+1}. And the transformation (1.1) may be always represented by the eouations

(1.4) $$z^{\alpha'}=z^{\alpha'}(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty}),\quad z^{\bar{\alpha}'}=z^{\bar{\alpha}'}(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty}),$$

$$z^{\infty'}=z^{\infty'}(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty});$$

$$\begin{vmatrix} \dfrac{\partial z^{\alpha'}}{\partial z^{\alpha}} & \dfrac{\partial z^{\alpha'}}{\partial z^{\bar{\alpha}}} & \dfrac{\partial z^{\alpha'}}{\partial z^{\infty}} \\ \dfrac{\partial z^{\bar{\alpha}'}}{\partial z^{\alpha}} & \dfrac{\partial z^{\bar{\alpha}'}}{\partial z^{\bar{\alpha}}} & \dfrac{\partial z^{\bar{\alpha}'}}{\partial z^{\infty}} \\ \dfrac{\partial z^{\infty'}}{\partial z^{\alpha}} & \dfrac{\partial z^{\infty'}}{\partial z^{\bar{\alpha}}} & \dfrac{\partial z^{\infty'}}{\partial z^{\infty}} \end{vmatrix} \neq 0.$$

Now, if it is possible to choose a coordinate neighborhood system, in such that, in the domain $U\cap U'$ of two coordinate neighborhoods $U(z^{i})$, $U'(z^{i'})$, it holds

(1.5) $$z^{\alpha'}=z^{\alpha'}(z^{\alpha}),\quad z^{\bar{\alpha}'}=z^{\bar{\alpha}'}(z^{\bar{\alpha}}),\quad z^{\infty'}=z^{\infty'}(z^{\infty}),$$

$$\left|\frac{\partial z^{\alpha'}}{\partial z^{\alpha}}\right|\cdot\left|\frac{\partial z^{\bar{\alpha}'}}{\partial z^{\bar{\alpha}}}\right|\cdot\frac{\partial z^{\infty'}}{\partial z^{\infty}}\neq 0,$$

we say that the manifold X_{2n+1} admits an ***adjoint complex structure*** and we call X_{2n+1} an ***adjoint complex manifold.***

From (1.5), since $\left|\dfrac{\partial z^{\alpha'}}{\partial z^{\alpha}}\right|\cdot\left|\dfrac{\partial z^{\bar{\alpha}'}}{\partial z^{\bar{\alpha}}}\right|>0$, the manifold is orientable, if and only if $\dfrac{\partial z^{\infty'}}{\partial z^{\infty}}>0.$

We consider the components of a tensor T^{h}_{i}, i.e.,

$$T^{h}_{i}=(T^{\alpha}_{\beta},\ T^{\bar{\alpha}}_{\beta},\ \cdots\cdots,\ T^{\infty}_{\infty}).$$

For coordinate transformation (1.5), they hold

(1.6) $$T^{h'}_{i'}=\frac{\partial z^{h'}}{\partial z^{h}}\frac{\partial z^{i}}{\partial z^{i'}}T^{h}_{i}.$$

From this and (1.5), $(T^{\alpha}_{\beta},\cdots\cdots,0)$, $(0,T^{\bar\alpha}_{\bar\beta},\cdots\cdots,0)$, $\cdots\cdots$, $(0,\cdots\cdots,0,T^{\infty}_{\infty})$ are the components of tensors. And from (1.6), we get

(1.7) $$\overline{T}^{\bar h'}_{\bar i'}=\frac{\partial z^{h'}}{\partial z^{h}}\frac{\partial z^{i}}{\partial z^{i'}}\overline{T}^{\bar h}_{\bar i},$$

where, if h takes the indices α, $\bar\alpha$, ∞, $\bar h$ takes $\bar\alpha$, α, ∞ respectively, and thus, if T^{h}_{i} are components of tensor, then $\overline{T}^{\bar h}_{\bar i}$ are the components of the tensor. We call $\overline{T}^{\bar h}_{\bar i}$ the *adjoint conjugate* of T^{h}_{i}, and if $T^{h}_{i}=\overline{T}^{\bar h}_{\bar i}$, we say that T^{h}_{i} is *adjoint self-conjugate*. If T^{h}_{i} is adjoint self-conjugate, then the component T^{∞}_{∞} is real.

Now, let us assume that our adjoint complex manifold admits a Riemanian metric

(1.8) $$ds^2=g_{ji}dz^jdz^i,$$

where symmetric tensor g_{ji} is adjoint self-conjugate and satisfies

(1.9) $$g_{ji}=\begin{pmatrix} 0 & g_{\beta\bar\alpha} & 0 \\ g_{\bar\beta\alpha} & 0 & 0 \\ 0 & 0 & g_{\infty\infty} \end{pmatrix}.$$

Then the metric form (1.8) can be written in the form

(1.10) $$ds^2=2g_{\beta\bar\alpha}dz^{\beta}dz^{\bar\alpha}+g_{\infty\infty}dz^{\infty}dz^{\infty}.$$

Of cause, this is real since $ds^2=\overline{ds^2}$. We call this metric satisfying (1.9) an *adjoint Hermitian metric* and the adjoint complex manifold with the adjoint Hermitian metric an *adjoint Hermitian manifold.*

§2. The Christoffel symbol of the adjoint Hermitian manifold.

In the adjoint Hermitian manifold X_{2n+1} with the metric (1.9) and (1.10), let us calculate the contravariant fundamental tensor g^{ji}. Then we have the compo nents of g^{ji} as follows:

(2.1) $$g^{ji}=\begin{pmatrix} 0 & g^{\beta\bar\alpha} & 0 \\ g^{\bar\beta\alpha} & 0 & 0 \\ 0 & 0 & g^{\infty\infty} \end{pmatrix},$$

(2.2) $$g^{\gamma\bar{\beta}}g_{\bar{\beta}\alpha}=\delta^{\gamma}_{\alpha},\quad g^{\bar{\gamma}\beta}g_{\beta\bar{\alpha}}=\delta^{\bar{\gamma}}_{\bar{\alpha}},\quad g^{\infty\infty}g_{\infty\infty}=1,$$

and g^{ji} is adjoint self-conjugate.

Next, let us calculate the Christoffel symbol

(2.3) $$\Gamma^{h}_{ji}=\frac{1}{2}g^{hr}(\partial_j g_{ir}+\partial_i g_{jr}-\partial_r g_{ji}).$$

Then we obtain the components as follows:

(2.4) $$\Gamma^{\alpha}_{\gamma\beta}=\frac{1}{2}g^{\alpha\bar{\tau}}(\partial_\gamma g_{\beta\bar{\tau}}+\partial_\beta g_{\gamma\bar{\tau}}),$$
$$\Gamma^{\alpha}_{\bar{\gamma}\beta}=\frac{1}{2}g^{\alpha\bar{\tau}}(\partial_{\bar{\gamma}} g_{\beta\bar{\tau}}-\partial_{\bar{\tau}} g_{\bar{\gamma}\beta}),$$
$$\Gamma^{\infty}_{\bar{\gamma}\beta}=-\frac{1}{2}g^{\infty\infty}\partial_\infty g_{\bar{\gamma}\beta},\qquad \Gamma^{\alpha}_{\infty\beta}=\frac{1}{2}g^{\alpha\bar{\tau}}\partial_\infty g_{\beta\bar{\tau}},$$
$$\Gamma^{\alpha}_{\infty\infty}=\frac{1}{2}g^{\alpha\bar{\tau}}\partial_{\bar{\tau}} g_{\infty\infty},\qquad \Gamma^{\infty}_{\beta\infty}=\partial_\beta\log\sqrt{g_{\infty\infty}},$$
$$\Gamma^{\infty}_{\infty\infty}=\partial_\infty\log\sqrt{g_{\infty\infty}},\qquad \Gamma^{\alpha}_{\bar{\gamma}\bar{\beta}}=\Gamma^{\alpha}_{\infty\bar{\beta}}=\Gamma^{\infty}_{\gamma\beta}=0,$$

and the values of the other components are given by symmetric and adjoint self-conjugate properties.

From the equation of the coordinate transformation

(2.5) $$\Gamma^{h'}_{j'i'}=\frac{\partial z^{h'}}{\partial z^{h}}\left(\frac{\partial z^{j}}{\partial z^{j'}}\frac{\partial z^{i}}{\partial z^{i'}}\Gamma^{h}_{ji}+\frac{\partial^2 z^{h}}{\partial z^{j'}\partial z^{i'}}\right),$$

If the (h', i', j') takes of the indices excepting $(\alpha', \beta', \gamma')$, $(\bar{\alpha}', \bar{\beta}', \bar{\gamma}')$ and $(\infty', \infty', \infty')$, then, since it holds

$$\frac{\partial z^{h'}}{\partial z^{h}}\frac{\partial^2 z^{h}}{\partial z^{j'}\partial z^{i'}}=0,$$

the condition

(2.6) $$\Gamma^{h}_{ji}=0,\ \text{excepting}\ \Gamma^{\alpha}_{\gamma\beta},\ \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}}\ \text{and}\ \Gamma^{\infty}_{\infty\infty},$$

is invariant under the coordinate transformation (1.5). And from (2.4), (2.6) is equivalent to

(2.7) $$\partial_\gamma g_{\beta\bar{\alpha}}=\partial_\beta g_{\gamma\bar{\alpha}},\qquad (\partial_{\bar{\gamma}} g_{\bar{\beta}\alpha}=\partial_{\bar{\beta}} g_{\bar{\gamma}\alpha}),$$
$$\partial_\infty g_{\beta\bar{\tau}}=0,$$
$$\partial_\tau g_{\infty\infty}=0,\qquad (\partial_{\bar{\tau}} g_{\infty\infty}=0).$$

The condition (2.6) or (2.7) is called *adjoint Kaehlerian condition*, and if an adjoint Hermitian manifold X_{2n+1} admits this adjoint Kaehlerian condition, we call this manifold X_{2n+1} an ***adjoint Kaehlerian manifold.***

From (2.7), we obtain the following:

THEOREM 1. *On an adjoint Kaelerian manifold, for the metric tensor, it holds that* $g_{\beta\bar{\alpha}}$ *depend only upon* z^{α} *and* $z^{\bar{\alpha}}$ *and* $g_{\infty\infty}$ *only upon* z^{∞}.

On the adjoint Kaehlerian manifold, the Christoffel symbols are represented by

$$\Gamma^{\alpha}_{\gamma\beta} = g^{\alpha\bar{\tau}}\, \partial_{\gamma}\, g_{\beta\bar{\tau}},\ \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}} = g^{\bar{\alpha}\tau}\, \partial_{\bar{\gamma}}\, g_{\bar{\beta}\tau},\ \Gamma^{\infty}_{\infty\infty} = \partial_{\infty} \log \sqrt{g_{\infty\infty}}, \tag{2.8}$$

and the others are zero.

§ 3. The curvature tensor in the adjoint Kaehlerian manifold.

In the adjoint Kaehlerian manifold X_{2n+1}, we calculate the Riemannian curvature tensor

$$R_{kji}{}^{h} = \partial_{k}\Gamma^{h}_{ji} - \partial_{j}\Gamma^{h}_{ki} + \Gamma^{h}_{ks}\Gamma^{s}_{ji} - \Gamma^{h}_{js}\Gamma^{s}_{ki}. \tag{3.1}$$

Form (2.8), we have

$$\begin{aligned} &R_{kj\bar{\beta}}{}^{\alpha} = 0, && R_{kj\beta}{}^{\bar{\alpha}} = 0, \\ &R_{kj\infty}{}^{\alpha} = 0, && R_{kj\beta}{}^{\infty} = 0, \\ &R_{kj\infty}{}^{\bar{\alpha}} = 0, && R_{kj\bar{\beta}}{}^{\infty} = 0, \end{aligned} \tag{3.2}$$

and for

$$R_{kjih} = R_{kji}{}^{s}\, g_{sh},$$

we obtain

$$\begin{aligned} &R_{kj\beta\alpha} = 0, && R_{kj\bar{\beta}\bar{\alpha}} = 0, \\ &R_{kj\beta\infty} = 0, && R_{kj\bar{\beta}\infty} = 0. \end{aligned} \tag{3.3}$$

Also

$$R_{kjih} = R_{ihkj}$$

implies

$$\begin{aligned} &R_{\beta\alpha kj} = 0, && R_{\bar{\beta}\bar{\alpha}kj} = 0, \\ &R_{\beta\infty kj} = 0, && R_{\bar{\beta}\infty kj} = 0. \end{aligned} \tag{3.4}$$

And from (2.4) and THEOREM 1, we obtain

$$R_{\delta\infty\infty\infty}{}^{\infty} = \partial_{\delta}\,\Gamma^{\infty}_{\infty\infty} = 0, \tag{3.5}$$

$$R_{\delta\infty\infty}{}^{\infty} = \partial_\delta \Gamma^{\infty}_{\infty\infty} = 0.$$

Then, from (3.2)∼(3.5), we can see that only the components of the form

$$(3.6) \qquad R_{\delta\gamma\beta}{}^{\alpha}, \quad R_{\delta\bar{\gamma}\bar{\beta}}{}^{\bar{\alpha}}, \quad R_{\delta\bar{\gamma}\beta}{}^{\alpha}, \quad R_{\bar{\delta}\gamma\beta}{}^{\alpha},$$

and

$$(3.7) \qquad R_{\delta\bar{\gamma}\beta\bar{\alpha}}, \quad R_{\bar{\delta}\gamma\bar{\beta}\alpha}, \quad R_{\bar{\delta}\gamma\beta\bar{\alpha}}, \quad R_{\delta\bar{\gamma}\bar{\beta}\alpha},$$

can be different from zero. And they are represented by

$$(3.8) \qquad R_{\bar{\delta}\gamma\beta}{}^{\alpha} = \partial_{\bar{\delta}}\, \Gamma^{\alpha}_{\gamma\beta}, \quad R_{\delta\bar{\gamma}\bar{\beta}}{}^{\bar{\alpha}} = \partial_\delta\, \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}}.$$

If $\Gamma^{\alpha}_{\gamma\beta}$ depend only upon z^{α}, then $\Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}}$ depend only upon $z^{\bar{\alpha}}$ since $\Gamma^{\alpha}_{\gamma\beta}$ are adjoint self-conjugate, and thus we obtain $R_{kji}{}^{h} = 0$, and hence we have the following:

THEOREM 2. *In the adjoint Kaehlerian manifold, if* $\Gamma^{\alpha}_{\gamma\beta}$ $\left(or\ \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}}\right)$ *depend only upon* z^{α} *(or* $z^{\bar{\alpha}}$*), then the manifold is flat.*

For the Ricci tensor R_{ji}, from (3.6), we have

$$(3.9) \qquad R_{\gamma\beta} = 0, \quad R_{\bar{\gamma}\bar{\beta}} = 0, \quad R_{\gamma\infty} = 0, \quad R_{\bar{\gamma}\infty} = 0, \quad R_{\infty\infty} = 0,$$

$$R_{\bar{\gamma}\beta} = R_{i\bar{\gamma}\beta}{}^{i} = R_{\alpha\bar{\gamma}\beta}{}^{\alpha} = -\,\partial_{\bar{\gamma}}\Gamma^{\alpha}_{\alpha\beta} = -\,\partial_{\bar{\gamma}}\,\partial_\beta \log \sqrt{g}.$$

And since it holds that

$$(3.10) \qquad g = |g_{ji}| = g_{\infty\infty}|g_{\beta\bar{\alpha}}|^2,$$

and $g_{\infty\infty}$ depends only upon z^{∞}, we have

$$(3.11) \qquad R_{\bar{\gamma}\beta} = -\,\partial_{\bar{\gamma}}\,\partial_\beta \log |g_{\beta\bar{\alpha}}|,$$

and the others are all zero.

§ 4. The structure of almost adjoint complex manifold.

In an adjoint complex manifold, there exist three mixed tensor F_i^h, G_i^h and H_i^h which have the numerical components

$$(4.1) \quad (F_i^h) = \begin{pmatrix} i\delta^{\alpha}_{\beta} & 0 & 0 \\ 0 & -i\delta^{\bar{\alpha}}_{\bar{\beta}} & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad (G_i^h) = \begin{pmatrix} -i\delta^{\alpha}_{\beta} & 0 & 0 \\ 0 & i\delta^{\bar{\alpha}}_{\bar{\beta}} & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$(H_i^h)=\begin{pmatrix} i\delta_\beta^\alpha & 0 & 0 \\ 0 & i\delta_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & -i \end{pmatrix}, \qquad (i=\sqrt{-1})$$

in all coordinate system (1.5), and which satisfy

$$(4.2) \qquad F_i^h F_j^i = -A_j^h,\quad G_i^h G_j^i = -A_j^h,\quad H_i^h H_j^i = -A_j^h$$

$$(4.3) \qquad F_i^h + G_i^h + H_i^h = iA_i^h,$$

$$(4.4) \qquad F_i^h G_j^i + G_i^h H_j^i + H_i^h F_j^i = A_j^h,$$

where A_j^h is unit tensor. Form (4.3), we can see that, if two among three tensors in (4.1) are given, then another is determined by their two tensors.

Let us put

$$(4.5) \qquad B_i^h = \frac{1}{2}(A_i^h + iG_i^h),\quad C_i^h = \frac{1}{2}(A_i^h + iF_i^h),$$

$$D_i^h = \frac{1}{2}(A_i^h + iH_i^h),$$

$$\bar B_i^h = \frac{1}{2}(A_i^h - iG_i^h),\quad \bar C_i^h = \frac{1}{2}(A_i^h - iF_i^h),$$

$$\bar D_i^h = \frac{1}{2}(A_i^h - iH_i^h).$$

Then their components are represented by

$$(4.6) \qquad (B_i^h)=\begin{pmatrix} \delta_\beta^\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad (C_i^h)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad (D_i^h)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(\bar B_i^h)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad (\bar C_i^h)=\begin{pmatrix} \delta_\beta^\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad (\bar D_i^h)=\begin{pmatrix} \delta_\beta^\alpha & 0 & 0 \\ 0 & \delta_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If, in a $(2n+1)$-dimensional differentiable manifold X_{2n+1} of class C^∞, there exist three mixed tensors F_i^h, G_i^h and H_i^h satisfying

$$(4.6) \qquad F_i^h F_j^i = -A_j^h,\quad G_i^h G_j^i = -A_j^h,\quad H_i^h H_j^i = -A_j^h,$$

$$(4.7) \qquad F_i^h G_j^i + G_i^h H_j^i + H_i^h F_j^i = A_j^h,$$

we say that the manifold admits an *almost adjoint complex structure* and we call such a manifold an *almost adjoint complex manifold*.

In the almost adjoint complex malifold, if there exists a coordinate system (1.4) with respect to which the two tensors among F_i^h, G_i^h and H_i^h have the com-ponents as (4.1), then the manifold is an adjoint complex manifold. In fact, in a domain in which two such coordinate systems (z^i), $(z^{i'})$ are valid, we have, for F_i^h and G_i^h,

$$(4.9)\qquad \frac{\partial z^{i'}}{\partial z^i} F_{i'}^{h'} = \frac{\partial z^{h'}}{\partial z^h} F_i^h, \quad \frac{\partial z^{i'}}{\partial z^i} G_{i'}^{h'} = \frac{\partial z^{h'}}{\partial z^h} G_i^h,$$

from which we obtain

$$\frac{\partial z^{\beta'}}{\partial z^{\overline{\alpha}}} = 0, \quad \frac{\partial z^{\beta'}}{\partial z^{\infty}} = 0, \quad \frac{\partial z^{\overline{\beta}'}}{\partial z^{\alpha}} = 0, \quad \frac{\partial z^{\overline{\beta}'}}{\partial z^{\infty}} = 0, \quad \frac{\partial z^{\infty'}}{\partial z^{\alpha}} = 0, \quad \frac{\partial z^{\infty'}}{\partial z^{\overline{\alpha}}} = 0,$$

and thus, it follows that

$$z^{\alpha'} = z^{\alpha'}(z^{\alpha}), \quad z^{\overline{\alpha}'} = z^{\overline{\alpha}'}(z^{\overline{\alpha}}), \quad z^{\infty'} = z^{\infty'}(z^{\infty}).$$

In this case, we say that the almost adjoint complex structure *is induced by* an adjoint complex structure.

Next, we find a condition that an almost adjoint complex manifold admits an adjoint complex structure.

If an almost adjoint complex manifold with structure tensors F_i^h, G_i^h, H_i^h ad-mits an adjoint complex structure, then

$$(4.10)\qquad B_i^h dz^i = 0, \quad C_i^h dz^i = 0, \quad D_i^h dz^i = 0$$

are completely integrable. Conversely, let us assume that (4.10) are completely integrable, where B_i^h, C_i^h, D_i^h are defined in the same form by F_i^h, G_i^h, H_i^h, and denote the solutions of (4.10) by

$$z^{\alpha'} = z^{\alpha'}(z^i) = \text{const.}, \quad z^{\overline{\alpha}'} = z^{\overline{\alpha}'}(z^i) = \text{const.}, \quad z^{\infty'} = z^{\infty'}(z^i) = \text{const.}$$

respectively. Solving for z^i, we have

$$z^h = z^h(z^{\alpha'}, z^{\overline{\alpha}'}, z^{\infty'}),$$

and thus,

$$dz^h = \frac{\partial z^h}{\partial z^{\alpha'}} dz^{\alpha'} + \frac{\partial z^h}{\partial z^{\overline{\alpha}'}} dz^{\overline{\alpha}'} + \frac{\partial z^h}{\partial z^{\infty'}} dz^{\infty'}.$$

From $dz^{\alpha'}=0$, $dz^{\overline{\alpha}'}=0$, $dz^{\infty'}=0$, since dz^h satisfies (4.10) respectively, we get

$$(4.11)\qquad \frac{\partial z^h}{\partial z^{\overline{\alpha}'}} = -iG_i^h \frac{\partial z^i}{\partial z^{\overline{\alpha}'}}, \quad \frac{\partial z^h}{\partial z^{\infty'}} = -iG_i^h \frac{\partial z^i}{\partial z^{\infty'}},$$

$$\frac{\partial z^h}{\partial z^{\alpha'}} = -iF_i^h\frac{\partial z^i}{\partial z^{\alpha'}}, \quad \frac{\partial z^h}{\partial z^{\infty'}} = -iF_i^h\frac{\partial z^i}{\partial z^{\infty'}},$$

$$\frac{\partial z^h}{\partial z^{\alpha'}} = -iH_i^h\frac{\partial z^i}{\partial z^{\alpha'}}, \quad \frac{\partial z^h}{\partial z^{\bar\alpha'}} = -iH_i^h\frac{\partial z^i}{\partial z^{\bar\alpha'}},$$

from which we have

$$(4.12) \qquad (G_i^h)=\begin{pmatrix} \sigma_\beta^\alpha & 0 & 0 \\ 0 & i\delta_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & i \end{pmatrix}, \quad (F_i^h)=\begin{pmatrix} i\delta_\beta^\alpha & 0 & 0 \\ 0 & \tau_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$(H_i^h)=\begin{pmatrix} i\delta_\beta^\alpha & 0 & 0 \\ 0 & i\delta_{\bar\beta}^{\bar\alpha} & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

From (4.8), we obtain

$$(4.13) \qquad \sigma_\beta^\alpha = -i\delta_\beta^\alpha, \quad \tau_{\bar\beta}^{\bar\alpha} = -i\delta_{\bar\beta}^{\bar\alpha}, \quad \lambda = -i,$$

which shows that F_i^h, G_i^h and H_i^h have the components (4.1) with respect to the coordinate system $(z^{\alpha'}, z^{\bar\alpha'}, z^{\infty'})$. Thus we have the following:

THEOREM 3. *In order that an almost adjoint complex manifold admits an adjoint complex structure, it is necessary and sufficient that* (4.10) *are completely integrable.*

§ 5. Almost adjoint Hermitian manifold.

In adjoint complex manifold, let us take three operators O, O^* and O^{**} such that

$$(5.1) \qquad O: O_{ji}^{ml} = \frac{1}{4}(\bar B_j^m C_i^l + C_j^m \bar B_i^l + B_j^m \bar C_i^l + \bar C_j^m B_i^l) = A_j^m A_i^l + \frac{1}{2}(G_j^m F_i^l + F_j^m G_i^l),$$

$$(5.2) \qquad O^*: O^*{}_{ji}^{ml} = \frac{1}{4}(\bar C_j^m D_i^l + D_j^m \bar C_i^l + C_j^m \bar D_i^l + \bar D_j^m C_i^l) = A_j^m A_i^l + \frac{1}{2}(F_j^m H_i^l + H_j^m F_i^l),$$

$$(5.3) \qquad O^{**}: O^{**}{}_{ji}^{ml} = \frac{1}{4}(\bar D_j^m B_i^l + B_j^m \bar D_i^l + D_j^m \bar B_i^l + \bar B_j^m D_i^l) = A_j^m A_i^l + \frac{1}{2}(H_j^m G_i^l + G_j^m H_i^l).$$

Then, for any covariant tensor g_{ji}, they are operated respectively by the forms

(5.4) $$(O_{ji}^{ml}\, g_{ml}) = \begin{pmatrix} 2g_{\beta\alpha} & 0 & g_{\beta\infty} \\ 0 & 2g_{\bar\beta\bar\alpha} & g_{\bar\beta\infty} \\ g_{\infty\alpha} & g_{\infty\bar\alpha} & 0 \end{pmatrix},$$

(5.5) $$(O^{*}{}_{ji}^{ml}\, g_{ml}) = \begin{pmatrix} 0 & g_{\beta\bar\alpha} & g_{\beta\infty} \\ g_{\bar\beta\alpha} & 2g_{\bar\beta\bar\alpha} & 0 \\ g_{\infty\alpha} & 0 & 2g_{\infty\infty} \end{pmatrix},$$

(5.6) $$(O^{**}{}_{ji}^{ml}\, g_{ml}) = \begin{pmatrix} 2g_{\beta\alpha} & g_{\beta\bar\alpha} & 0 \\ g_{\bar\beta\alpha} & 0 & g_{\bar\beta\infty} \\ 0 & g_{\infty\bar\alpha} & 2g_{\infty\infty} \end{pmatrix},$$

from which we have

(5.7) $$O\cdot O^{*}\cdot O^{**}(g) = (0).$$

If an almost adjoint complex manifold has a Riemannian metric $ds^2 = g_{ji}dz^j dz^i$ satisfying

(5.8) $$O_{ji}^{ml}\, g_{ml} = 0,$$

or

(5.9) $$g_{ji} = -\, g_{ml}\, F^{m}_{(j}\, G^{l}_{i)},$$

where the bracket denotes the symmetric part of the indices, then the manifold is called an *almost adjoint Hermitian manifold* and we call this metric an *almost adjoint Hermitian metric.*

From this definition, we have the following instantly:

THEOREM 4. *In an adjoint complex manifold* X_{2n+1}, *if it admits an almost adjoint Hermitian metric, then it is an adjoint Hermitian manifold, and conversely.*

In fact, it follows that, from (5.4) and (5.8),

$$g_{\beta\alpha} = g_{\bar\beta\bar\alpha} = 0, \qquad g_{\beta\infty} = g_{\bar\beta\infty} = 0,$$

and, since ds^2 is real, g_{ji} is adjoint self-conjugate.

And, we obtain the following theorem is valid:

THEOREM 5. *In an adjoint complex manifold, it is always possible to define an adjoint Hermitian metric.*

In fact, let a_{ji} be a tensor which defines a Riemannian metric in an adjoint complex manifold, and let us put

$$g_{ji} = O^{*ml}_{ji} O^{**ts}_{ml}\, a_{ts}, \tag{5.10}$$

then g_{ji} defines another positive definite Riemannian metric and satisfies (5.8).

§ 6. Almost adjoint Kaehlerian manifold.

In an almost adjoint Hermitian manifold X_{2n+1}, we define the tensors G_{ji}, F_{ji}, and H_{ji} as

$$G_{ji} = g_{jh}G^h_i,\quad F_{ji} = g_{jh}F^h_i,\quad H_{ji} = g_{jh}H^h_i, \tag{6.1}$$

and G_{kji}, F_{kji} and H_{kji} as

$$\begin{aligned} G_{kji} &= \nabla_k G_{ji} + \nabla_j G_{ik} + \nabla_i G_{kj}, \\ F_{kji} &= \nabla_k F_{ji} + \nabla_j F_{ik} + \nabla_i F_{kj}, \\ H_{kji} &= \nabla_k H_{ji} + \nabla_j H_{ik} + \nabla_i H_{kj}, \end{aligned} \tag{6.2}$$

In an adjoint Hermitian manifold, let us calculate the above tensors. Form (1.9) and (4.1), we have

$$(G_{ji}) = \begin{pmatrix} 0 & ig_{\beta\bar{\alpha}} & 0 \\ -ig_{\bar{\beta}\alpha} & 0 & 0 \\ 0 & 0 & ig_{\infty\infty} \end{pmatrix},\quad (F_{ji}) = \begin{pmatrix} 0 & -ig_{\beta\bar{\alpha}} & 0 \\ ig_{\bar{\beta}\alpha} & 0 & 0 \\ 0 & 0 & ig_{\infty\infty} \end{pmatrix}, \tag{6.3}$$

$$(H_{ji}) = \begin{pmatrix} 0 & ig_{\beta\bar{\alpha}} & 0 \\ ig_{\bar{\beta}\alpha} & 0 & 0 \\ 0 & 0 & -ig_{\infty\infty} \end{pmatrix}.$$

And we can see that for the tensors G_{kji}, F_{kji} and H_{kji}, their essential components are

(6.4) $$G_{\gamma\beta\alpha},\ G_{\gamma\beta\bar{\alpha}},\ G_{\gamma\beta\infty},\ G_{\gamma\bar{\beta}\bar{\alpha}},\ G_{\gamma\bar{\beta}\infty},$$
$$G_{\gamma\infty\infty},\ G_{\bar{\gamma}\bar{\beta}\bar{\alpha}},\ G_{\bar{\gamma}\bar{\beta}\infty},\ G_{\gamma\infty\infty},\ G_{\infty\infty\infty},$$

and F_{kji} and H_{kji} are similar. In consequence that G_{kji} are represented by

(6.5) $$G_{kji} = \partial_k G_{ji} + \partial_j G_{ik} + \partial_i G_{kj} - \Gamma^m_{kj}(G_{mi} + G_{im}) - \Gamma^m_{ji}(G_{mk} + G_{km}) - \Gamma^m_{ik}(G_{mj} + G_{jm}),$$

and the others are similar. Using of (2.4) and (6.3) we obtain the followings:

(6.7)
$$G_{\gamma\beta\alpha} = 0, \qquad G_{\gamma\beta\bar{\alpha}} = i(\partial_\gamma g_{\beta\bar{\alpha}} - \partial_\beta g_{\gamma\bar{\alpha}})$$
$$G_{\gamma\beta\infty} = 0, \qquad G_{\gamma\bar{\beta}\bar{\alpha}} = i(\partial_{\bar{\alpha}} g_{\gamma\bar{\beta}} - \partial_{\bar{\beta}} g_{\gamma\bar{\alpha}}),$$
$$G_{\gamma\bar{\beta}\infty} = 2i\,\partial_\infty g_{\gamma\bar{\beta}}, \qquad G_{\gamma\infty\infty} = -i\,\partial_\gamma g_{\infty\infty},$$
$$G_{\bar{\gamma}\bar{\beta}\bar{\alpha}} = 0, \qquad G_{\bar{\gamma}\bar{\beta}\infty} = 0,$$
$$G_{\bar{\gamma}\infty\infty} = -i\,\partial_{\bar{\gamma}} g_{\infty\infty}, \qquad G_{\infty\infty\infty} = 0,$$

(6.8)
$$F_{\gamma\beta\alpha} = 0, \qquad F_{\gamma\beta\bar{\alpha}} = -i(\partial_\gamma g_{\beta\bar{\alpha}} - \partial_\beta g_{\gamma\bar{\alpha}}),$$
$$F_{\gamma\beta\infty} = 0, \qquad F_{\gamma\bar{\beta}\bar{\alpha}} = -i(\partial_{\bar{\alpha}} g_{\gamma\bar{\beta}} - \partial_{\bar{\beta}} g_{\gamma\bar{\alpha}}),$$
$$F_{\gamma\bar{\beta}\infty} = 2i\,\partial_\infty g_{\gamma\bar{\beta}}, \qquad F_{\gamma\infty\infty} = -i\,\partial_\gamma g_{\infty\infty},$$
$$F_{\bar{\gamma}\bar{\beta}\bar{\alpha}} = 0, \qquad F_{\bar{\gamma}\bar{\beta}\infty} = 0,$$
$$F_{\bar{\gamma}\infty\infty} = -i\,\partial_{\bar{\gamma}} g_{\infty\infty}, \qquad F_{\infty\infty\infty} = 0,$$

(6.9)
$$H_{\gamma\beta\alpha} = 0, \qquad H_{\gamma\beta\bar{\alpha}} = 0, \qquad H_{\gamma\beta\infty} = 0,$$
$$H_{\gamma\bar{\beta}\bar{\alpha}} = 0, \qquad H_{\gamma\bar{\beta}\infty} = 0, \qquad H_{\gamma\infty\infty} = i\,\partial_\gamma g_{\infty\infty},$$
$$H_{\bar{\gamma}\bar{\beta}\bar{\alpha}} = 0, \qquad H_{\bar{\gamma}\bar{\beta}\infty} = 0, \qquad H_{\bar{\gamma}\infty\infty} = i\,\partial_{\bar{\gamma}} g_{\infty\infty},$$
$$H_{\infty\infty\infty} = 0.$$

Form these and the adjoint Kaehlerian condition (2.7), we can see that $G_{kji}=0$ or $F_{kji}=0$ is necessary and sufficient in order that an adjoint Hermitian manifold be an adjoint Kaehlerian.

If, in an almost adjoint Hermitian manifold, it satisfies

(6.10) $$G_{kji}=0, \qquad F_{kji}=0,$$

we say that the manifold admits an *almost adjoint Kaehlerian structure* and we call it an *almost adjoint Kaehlerian manifold*. And thus we have the following:

THEOREM 6. *If an adjoint Hermitian manifold admits an almost adjoint Kaehlerian structure, then it is an adjoint Kaehlerian manifold, and conversely.*

§ 7. Diagram.

Summerizing the theorems in the present paper, we obtain the following diagram:

Almost adjoint complex manifold	⟶ $B^h_i dz^i=0$, $C^h_i dz^i=0$, $D^h_i dz^i=0$ are completely integrable.	Adjoint complex manifold
↓ $O^{ml}_{ji}\, g_{ml}=0$		↓ $O^{ml}_{ji}\, g_{ml}=0$
Almost adjoint Hermitian manifold	⟶ $B^h_i dz^i=0$, $C^h_i dz^i=0$, $D^h_i dz^i=0$ are completely integrable.	Adjoint Hermitian manifold
↓ $G_{kji}=0$, $F_{kji}=0$		↓ $G_{kji}=0$, $F_{kji}=0$
Almost adjoint Kaehlerian manifold	⟶ $B^h_i dz^i=0$, $C^h_i dz^i=0$, $D^h_i dz^i=0$ are completely integrable.	Adjoint Kaehlerian manifold

July, 1964

Mathematical Department
Kyungpook University
Taegu, Korea

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