◇ 解　　　説 ◇

JORDAN ALGEBRAS

徐　泰　日 (T. Suh)

1. It is the purpose of the present paper to give a brief survey and present status of Jordan algebras with their connected fields. *A non-associative* (not necessarily associative) *algebra* **a** over a field Φ is a vector space over Φ in which a bilinear composition is defined. That is, for any pair (x,y) of elements x and y of a vector space **a** there is associated with a product xy in **a** satisfying the following bilinearity conditions

(1) $x(y+z)=xy+xz$, $(y+z)x=yx+zx$

(2) $\alpha(xy)=(\alpha x)y=x(\alpha y)$, α in Φ.

The algebra **a** is said to be *associative* if for all x, y, z of **a** we have the associative law

(3) $x(yz)=(xy)z$.

A non-associative algebra **a** is called *an alternative algebra* if its multiplication satisfies the alternative law

(4) $x^2y=x(xy)$, $yx^2=(yx)x$.

A non-associative algebra **a** is called *a Lie* algebra if its multiplication satisfies the Lie conditions

(5) $x^2=0$, $(xy)z+(yz)x+(zx)y=0$.

The second of these is called *the Jacobi identity*. The general theory of Lie algebras has been developed extensively and applications of the theory are to be found in many other branches of mathematics. A full account of the theory may be found in the book [13] written by Jacobson. We define a *Jordan algebra* to be a non-associative algebra **a** whose multiplication satisfies the Jordan conditions

(6) $xy=yx$, $x^2(yx)=(x^2y)x$.

If an algebra satisfies one of the identities (4), (5), (6), so does any subalgebra or homomorphic image of it. If $\{\mathbf{a}_\alpha\}$ is a family of algebras satisfying one of the above, then so does the algebra $\Sigma_\alpha \oplus \mathbf{a}_\alpha$, the complete direct sum. In general, a class of algebras defined by a set of identities will be closed under the operation of taking subalgebra, homomorphic images and complete direct sums. Conversely, a special case of a result of G. Birkhoff [3] implies that any class of non-associative algebras that is closed under the operations of taking subalgebras, homomorphic images and complete direct sums can be defined by a set of identies.

2. It might be said that alternative algebras begin and end with the Cayley numbers since they were defined in order to study the Cayley numbers, which in turn are the only distinguished members of that class. We give the construction of a (generalized) Cayley algebra. Let Q be a (generalized) quaternion algebra and C the eight-dimensional vector space of elements of the form: $a+bl$, a, $b \epsilon Q$, l a symbol. Addition and scalar multiplication of these elements are defined in the usual way. Multiplication is defined by

$$(a+bl)(c+dl)=(ac+\mu\bar{d}b)+(da+b\bar{c})l$$

where μ is a non-zero element of the base field and $\bar{a}$ is the usual conjugate of the quaternion a. Then we have an alternative algebra which is not associative. Under suitable conditions on Q and μ, C is a division algebra, an algebra whose non-zero elements have inverses.

It is proved by R. H. Bruck and E. Kleinfeld [4] that the only alternative division algebras which are not associative are the Cayley algebras. This is also proved by L. A. Skornyakov [19] independently. Instead of going into detail we list a few references for the convenience of the reader. For an introduction to some of the geometrical aspects of alternative division algebras, there is a paper by Bruck [5]. Some of the number theoretic aspects are treated by Coxeter [7] and Kaplansky [14]. General results about the structure of simple alternative algebras are due to Albert [2] and Kleinfeld [15]. The automorphisms of alternative algebras has been touched first by Zorn [22]. If the base field of the algebra is the field of complex numbers then the group of automorphisms is the Lie group G_2 in the Killing-Cartan clssification. Analogues of these groups for arbitrary fields have been defined by Chevalley [6]. In this connection the reader is referred to the paper [10] by Jacobson.

3. We consider an associative algebra $\mathbf{a}$ over a field Φ of characteristic not two. By using the associative multiplication of elements of $\mathbf{a}$ written $x\cdot y$, let us define new compositions

(7) $[x, y]=x\cdot y-y\cdot x$

(8) $xy=\frac{1}{2}(x\cdot y+y\cdot x)$.

The vector space structures of $\mathbf{a}$ together with the new compositions $[x,y]$ and xy give us the Lie algebra $\mathbf{a}^-$ and the Jordan algebra $\mathbf{a}^+$ respectively.

In the theory of Lie algebras the Poincaré-Birkhoff-Witt theorem [13] says that every Lie algebra is isomorphic to a subalgebra of $\mathbf{a}^-$, $\mathbf{a}$ an associative algebra. However, in the theory of Jordan algebras we don't have the analogue of the Poincaré-Birkhoff-Witt theorem. A Jordan algebra J is said to be *special* if J is isomorphic to a subalgebra of $\mathbf{a}^+$, $\mathbf{a}$ associative. The Jordan algebras which are not special are called *exceptional Jordan algebras*. Every special Jordan algebra is characterized as a subalgebra of a J-algebra. By *a J-algebra* we mean a commutative algebra K (with product xy) having an additional binary operation $[x, y]$ defined on K which is bilinear, antisymmetric, and satisfies

$[x,y^2]=2[x,y]y$, $[x,[y,z]]=4\{(xy)z-(xz)y\}$.

Every J-algebra is of the form $\mathbf{a}^+$, $\mathbf{a}$ a suitable associative algebra.

The Wedderburn structure theorem for associative algebras has a Jordan algebra analogue. That is, any finite-dimensional semisimple Jordan algebra over a field of characteristic zero is a direct sum of ideals which are simple Jordan algebras. We consider the classification of simple Jordan algebras. It is well known that the problem of simple algebras can be reduced to the study of central simple algebras. There are four classes of central simple Jordan algebras among special Jordan algebras and only one class of exceptional Jordan algebras. That is, according to Albert [1], Jacobson [8] and Schafer [17] if J is central simple Jordan algebras over an arbitrary field Φ then J is one of the following types:

A. J is of the form $\mathbf{a}^+$, $\mathbf{a}$ central simple associative or of the form $H(\mathbf{a},\pi)=\{a\epsilon\mathbf{a} \mid a^\pi=a\}$ where $\mathbf{a}$ is simple with center, a separable quadratic extension of Φ and π isan involution. In the latter case, $J_P(=P\otimes_\Phi J)$ is $\mathbf{a}^+$ for a suitable extension P of Φ.

B-C. $J=H(\mathbf{a},\pi)=\{a\epsilon\mathbf{a} \mid a^\pi=a\}$, $\mathbf{a}$ a central simple associative algebra over Φ with the involution π. The enveloping associative algebra of J is $\mathbf{a}$. Also $\mathbf{a}_\Omega(=\Omega\otimes_\Phi\mathbf{a})$ is Ω_m for Ω the algebraic closure of Φ and the involution π can be taken to have one of the following two forms in Ω_m: $a^\pi=a'$ or $a^\pi=q^{-1}a'q$ where a' is the transpose of a and q is any non-singular skew symmetric matrix. Then H_Ω is either the set of symmetric matrices or the set of symplectic symmetric matrices $(q^{-1}a'q=a)$. In the first case J is of type B and in the second J is of type C.

D. Let M be a vector space over Φ equipped with a non-degenerate symmetric bilinear form (a, b). Consider the algebra $J=\Phi 1\oplus M$ determined by (a, b) by the rules that 1 is the identity of J, and for $a, b\,\epsilon\, M$ the product ab is given by $ab=(a, b)1$. Then the algebra J is a Jordan algebra which is central simple if dim $M>1$ and is called *the Jordan algebra of the bilinear form* (a, b) in M.

The algebras A-D are special. Besides these we have

E. The exceptional central simple Jordan algebras. These are 27 dimensional and if the base field is algebraically closed then such an algebra J is isomorphic to the algebra $H(C_3, \gamma)$ of 3×3 γ-hermitian (Cayley) matrices of C_3 relative to the product $xy=\frac{1}{2}(x\cdot y+y\cdot x)$.

By virtue of their exceptional character, the exceptional Jordan algebras are perhaps the most interesting of all Jordan algebras. The simple Lie algebras over an algebraically closed field fall into four classes and five exceptional algebras. These classes of Lie algebras have a definite relationship with the classes of Jordan algebras which mentioned above. Particularly the exceptional Lie algebras are closely related to the exceptional simple Jordan algebras. These relations are investigated by Chevalley, Freudenthal, Jacobson, Schafer, Springer and others. These investigations have led to the different interpretations for the new simple groups and a new model of Cayley planes coordinated by means of Cayley algebras. The general algebraic form of the Jordan-Freudenthal coordinatization of the planes has been given by Springer [20] and by using this certain linear groups of type E_6 over an arbitrary fields have been studied by Jacobson [11] and Suh [21]. There are still many unsolved problems in these branches to give ample challenge to us.

The representation theory for Jordan algebras which is comparable to its associative counterpart and has not been touched in our brief discussion may be found in the paper [9] of Jacobson. It is also worthwhile noting that there are unsolved problems in the existing theory of Jordan algebras and problems arising from the latest applications of Jordan algebras to geometry.

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