OPEN SUBGROUPS IN CHARACTER GROUPS

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This note is a study in locally compact abelian groups, and is concerned with orthogonal pairs defined by two groups. It refers to open subgroups in a group of an orthogonal pair and the character group of the other group.

To simplify our discussions, we recall some knowledge assumed as known, and also introduce some notations.

We say that two groups G and X form a PAIR, whenever there is given for every pair of elements of $x \in G$ and $\xi \in X$ a product $x\xi = \xi x$, continuous and distributive in both variables and whose values are in K. In a pair G and X we denote the elements of G by small italic letters: $x, y, \cdots$, and the elements of X by small Greek letters: $\xi, \eta, \cdots$.

Let H be a subset of G in a pair G and X. We call the set of all elements ξ of X for which $x\xi = O$ for every $x \in H$ the ANNIHILATOR $[X, H]$. Similarly with G, X interchanged. In particular, a pair G and X is called ORTHOGONAL if

$$[G, X] = \{O\} \text{ and } [X, G] = \{O\}.$$

We say that a group G has A COMPACT GENERATOR if there exists a nucleus V of G such that G is generated by V and $\overline{V}$ is compact.

A mapping f of a set G into a set X is called INJECTIVE if the condition $x \neq y$ (where $x, y \in G$) implies $f(x) \neq f(y)$.

Frequently used symbols.

C=An infinite cyclic group.

D=The additive real group.

K=The additive group of the reals mod 1

Z=A finite abelian group.

G_i=The subgroup $(O, \cdots, O, G_i, O, \cdots, O$ in a external direct sum G of group of a finite system $G_1, \cdots, G_n$.

x_i=An element $(O, \cdots, O, X_i, O, \cdots, O)$ of G_n.

Λ_k=A nucleus determined dy its radius k in K.

$W(F, \Lambda_k)$=The set of characters of a group G carrying a compact subset F of G into Λ_k.

R_y=A mapping: $\xi \to x\xi$ in a pair G and X, where $\xi \in X$ and x is a fixed element of G.

$K^a C^b D^c Z$=A direct sum of groups K, C, D, and Z, where the finite numbers a, b, anc c indicate their numbers of groups respectively. We call such a group the fundamental group.

If a group G has a compact generator, G is the direct sum of its maximal compact subgroup H_1 and the fundamental group H_2 isomorphic to $C^a D^b$.[1]

If a pair G and X is orthogonal and H is a subgroup of G, then H and $X/[X, H]$ are orthogonal under the following multiplication: if $x \in H$ and $\xi^* \in X/[X, H]$, then $x\xi^* = x\xi$ for $\xi \in \xi^*$.

In an orthogonal pair G and X, let G^* be the character group of X. Then there exists a natural homomorphism F: $x \to x^*$ of G into G^* such that $x^*(\xi) = x\xi$, where $\xi \in X$.

(LEMMA 1) Let G^* be a direct sum of a finite system $G_1, \cdots G_n$. Suppose that the group G and a group X form an orthogonal pair. Let G_i^* be the character group of $X/[X, G_i]$, g_i the natural homomorphism of X on $X/[X, G_i]$, and F_i the natural homomorphism of G_i into G_i^*. Then the Pontrjagin Duality Theorem holds in the pair G and X if $F_i(G_i) = G_i^* \cong G_i$ and the algebraic subgroup $\Sigma G_i^* g_i$ is closed in the character group G^* of X

PROOF. If $x_i^* \in G_i^*$, then the resultant $x_i^* g_i$

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of g_i and x_i^* is a character of X. If F is any compact subset of X and Λ_k any nucleus of X, we may choose as nuclear base for the subspace $G_i^*g_i$ of G^* the familly of all the sets $W^*(F,\Lambda_k)$ $\cap G_i^*g_i$. Let $V_i(g_i(F),\Lambda_k)$ be the nucleus of G_i^* determined by $g_i(F)$ and Λ_k. Then we have $W_i^*(F, \Lambda_k)\cap G_ig_i=V_i(g_i(F), \Lambda_k)g_i$. Let $W_i(F,\Lambda_k)$ $=V_i(g_i(F), \Lambda_k)g_i$.

We shall now show that every $F_i : G_i \to G_i^*$ induces an isomorphism of G_i onto $G_i^*g_i$. Define a mapping $f_i:G_i\to G_i^*g_i$ on G_i by $f_i(x_i)=F_i(x_i)g_i$ $=x_i^*g_i$. Then we have

$$f_i(x_i)(\xi)=F_i(x_i)g_i(\xi)=\xi x_i$$

for all $\xi\in X$. If x_i, $y_i \in G_i$, then

$$\begin{aligned} f_i(x_i+y_i)(\xi)&=\xi(x_i+y_i)\\ &=f_i(x_i)(\xi)+f_i(y_i)(\xi)\\ &=(f_i(x_i)+f_i(y_i))(\xi). \end{aligned}$$

It follows by the continuity of F_i that for a nucleus $V_i(g_i(F), \Lambda_k)$ there exists a nucleus U of G_i such that $F_i(U)(g_i(F))\subset\Lambda_k$. Then we have

$$f_i(x_i)(F)=F_i(x_i)(g_i(F))\subset\Lambda_k$$

for every $x_i\in U$. Hence f_i is a homomorphism of G_i into $G_i^*g_i$. Furthermore, the set H_i of all elements $x_i\in G_i$ for which $x_i^*g_i(\xi)=\xi g_i=0$ for all $\xi\in X$. Then we have $H_i=\{0\}$. It follows by the isomorphism F_i that for a nucleus U of G_i there exists a nucleus $Vi(Q, \Lambda_k)$ of G_i^* such that $V_i(Q, \Lambda_k)\subset F_i(U)$, where Q is a compact subset of $X/[X, G_i]$. Take a nucelus U' of X so that $\overline{U'}$ is compct. Since Q is contained in a set of finite unions of sets $g_i(\xi_i+U')$, where $\xi_i\in X$, we can construct a compact subset F of X for which $Q\subset g_i(F)$. Then we have $V_i(g_i(F), \Lambda_k)\subset V_i(Q, \Lambda_k)$, and so $V_i(g_i(F), \Lambda_k)\subset F_i(U)$, whene $W_i(F, \Lambda_k\subset f_i(U)$. Hence if follows that f_i is an isomorphism of G onto $G_i^*g_i$

Let $G^{**}=\Sigma G_i^*g_i$. Then

$$\zeta\in G_i^*g_i\cap(G_1^*g_1+\cdots+G_{i-1}^*g_{i-1}+G^*_{i+1}g_{i+1}+\cdots+G_n^*g_n),$$

implies $\zeta=x_i^*g_i$ and

$$\zeta=x_1^*g_1+\cdots+x^*_{i-1}g_{i-1}+x^*_{i+1}g_{i+1}+\cdots+x^*_ng_n.$$

For all $\xi\in X$, we have

$$x_i^*g_i(\xi)=x_1^*g_1(\xi)+\cdots+x^*_{i+1}g_{i+1}(\xi)+x^*_{i+1}g_{i+1}(\xi)+\cdots+x_n^*g_n(\xi),$$

and so

$$\xi x_i=\xi x_1+\cdots+\xi x_{i-1}+\xi x_{i+1}+\cdots\xi x_n.$$

This gives us the contradiction

$$x_i=x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n,$$

whence $\zeta=0$. Thus the represetation of G^{**} is unique.

Now define a mapping $f:x\to\Sigma x_i^*g_i$ on G by

$$(\Sigma x_i^*g_i)(\xi)=\Sigma x_i^*g_i(\xi)=\Sigma\xi x_i=\xi x,$$

where $x=\Sigma x_i \in X$. Then we have $f(x)=\Sigma f_i(x_i)$. Since every f_i is a isomorphic mapping, f is an isomorphism of G onto G^{**}. Furthermore, a character $\xi\in X$ of G^* implies

$$\xi(\Sigma x_i^*g_i)=(\Sigma x_i^*g_i)(\xi)=\xi x.$$

Because of the orthogonality and the closed subset $\Sigma G_i^*g_i$ we have

$$(X, G^{**})=[X, G]=\{0\}$$

and so

$$G^{**}=(G^*, \{0\})=G^*.$$

Hence $G^{**}=G^*$, and G is the character group of X. We can conclude that the Pontrjagin Duality Theorem holds in the pair G and X

(LEMMA 2) Let G be a group which has a compact generator, and $G=H_1+H_2$ (Ref. 1). Suppose that the roup G and a group X form an orthogonal pair. Let H_i^* be the character group of $X/[X, H_i]$ and g_i the natural homomorphism of X on $X/[X, H_i]$. Then the Pontrjagin Duality Theorem holds in the pair G and X if the algebraic subgoup $\Sigma H_i^*g_i$ is closed in the character group G^* of X and $X/[X, H_2]\cong K^aD^b$.

PROOF. Let F_1 be the natural homomorphism of H_1 into H_1^*. Refering LEMMA 1, it sufices to prove that $F_1(H_1)=H_1^*\cong H_1$. Since G has a compact generator, there exists a nucleus W of G for which

$$G=W\cup 2W\cup\cdots\cup\ nW\cup\cdots,$$

and $\overline{nW}$ is compact. Then the set $\overline{nW\cap H_1}$ is compact and

$$H_1=(W\cap H_1)\cup(2W\cap H_1)\cup\cdots.$$

Hence H_1 is the group of countable unions of its compact subsets, and so the continuous mapping F_1 is open. Moreover, for $x(\neq0)\in H_1$ there exists an element $\xi^*\in X/[X, H_1]$ such that $F_1(x)(\xi^*)$ $=\xi^*x\neq0$. This implies that the character $F_1(x)$

is not null. That is to say, the kernel of F is null. Hence F_1 is an isomorphic mapping of H_1 on a subgroup $F_1(H_1)$ of $H_1{}^*$. On the other hand, $F_1(H_1)$ is a closed subgroup of $H_1{}^*$ since a group which is a T_1-space is a Hausdorff space. Because of the orthogonality we have $(H_1, F_1(H_1)) = \{0\}$, whence $F_1(H_1)=(H_1{}^*, \{0\})=H_1{}^*$. Thus F_1 is an isomorphism of H_1 onto $H_1{}^*$, and the proof is finished.

(THEOREM) Let a pair G and X be orthogonal. Let H be an open subgroup G which has a compact generator, and $H=H_1+H_2$ (Ref. 1). Then there exists an open subgroup H^* in the character group G^* of X such that $H\cong H^*$ and $G/H\cong G^*/H^*$ if and only if the following condition are satisfied:

(P. 1). The algebraic subgroup $\Sigma H_i{}^*g_i$ is closed in G_*, where $H_i{}^*$ is the character group of $X/[X, H_i]$ and g_i is the natural homomorphism of X on $X/[X, H_i]$,

(P. 2). $X/[X, H_2]\cong K^aD^b$,

(P. 3). The set $[X, H]$ is compact,

(P. 4). For the pair G/H and $[X, H]$ the algebraic mapping: $x^*\to R_x{}^*(x^*\epsilon\, G/H)$ is injective.

PROOF. Suppose that a pair G and X satisfies the conditions of proposition. Let g be the natural homomorphism of X on $X/[X, H]$. For every H_i of H, we have $[X, H]\subset[X, H_i]$.
and

$$g([X, H_i])=[X/[X, H], H_i].$$

This implies

$$X/[X,H_i]\cong X/[X,H]/[X/[X,H], H_i].$$

Let $Z_i=X/[X,H]/[X/[X,H], H_i]$. Let $H_i{}^{**}$ be the character group of Z_i, and $g_i{}^*$ the natural homomorhpism of $X/[X, H_i]$ on Z_i. Considering the orthogonal pairs: H_i and $X/[X, H_i]$, H_i and Z_i, we obtain $H_i\cong H_i{}^*$ and $H_i\cong H_i{}^{**}$ (Ref. LEMMA 2). Then we have $H_i\cong H_i{}^*g_i$ and $H_i\cong H_i{}^{**}g_i{}^*$ (Ref. LEMMA 1), and so $\Sigma' H_i{}^*g_i \cong H\cong \Sigma H_i{}^{**}g_i{}^*$. For every $x\epsilon H$ there exists $x_i{}^*\epsilon H_i{}^*$ and $x_i{}^{**}\epsilon H_i{}^{**}$ such that for every $\xi\epsilon X$

$$(\Sigma x_i{}^*g_i)(\xi)=\Sigma x_i{}^*g_i(\xi)=\Sigma\xi x_i=\xi x=\xi^* x$$
$$=\Sigma x_i{}^{**}g_i{}^*(\xi^*)=(\Sigma x_i{}^{**}g_i{}^*)(\xi^*)$$
$$=(\Sigma x_i{}^{**}g_i{}^*)(\xi),\ \text{where}\ \xi^*=\xi+[X,H].$$

This implies $\Sigma H_i{}^*g_i=\Sigma H_i{}^{**}g_i{}^*$ in G^*. On the other hand, the set $\Sigma H_i{}^{**}g_i{}^*$ is a subset of the character group $(G^*, [X, H])$ of $X/[X, H]$. Now let $H^*=(G^*, [X, H]$.) Then the set $\Sigma H_i{}^{**}g_i{}^*$ is closed in H^*. We conclude from this with the help of LEMMA 2 that each of the groups H and $X/[X, H]$ is the character group of the other.

The orthogonality of the pair G and X implies $[(X, H), G/H]=\{0\}$. Since $x^*\neq y^*$ (where x^*, $y^*\epsilon\, G/H$) implies $R_x{}^*\neq R_y{}^*$ (See (P. 4)), we have $[G/H, (X, H)]=\{0\}$. Hence the compact group $[X, H]$ and the discrete group G/H form an orthogonal pair, which concludes that each of two groups $[X, H]$ and G/H is the character group of the other.

By the character group G^*/H^* of $[X, H]$, we have $G/H\cong G^*/H^*$. By the character group H^* of $X/[X, H]$ we have $H\cong H^*$. Take a character $x^*\epsilon X$ so that $x^*([X,H])\cong\Lambda_k$. Then $x^*([X.H])=0$ and so $x^*\epsilon H^*$. This implies $H^*=W^*([X,H], \Lambda_k)$, that is to say, H^* is an open subgroup of G^*.

We shall show that the four condition of proposition are necessary. Let φ be the isomorphism of H onto H^*. Let V be a nucleus, which generates H, and whose closure is compact. Then $V^*=\varphi(V)$ is a nucleus which generates H^*. Since $\varphi(\overline{V})$ is compact and $\overline{V^*}\subset\varphi(\overline{V})$, the closure $\overline{V^*}$ is compact. It then follows that H^* has a compact generator and it is expressible in the form $H^*=H_1{}^*+H_2{}^*$. We see that $H_i\cong H_i{}^*$. In fact, the compact subgroup $\varphi(H_1)$ is contained in H^*, and also H_1 contains the compact subgroup $\varphi^{-1}(H_1{}^*)$, whence $\varphi(H_1)=H_1{}^*$, i.e., $H_1\cong H_1{}^*$. We have $H/H_1\cong H^*/H_1{}^*$, and so $H_2\cong H_2{}^*\cong C^aD^b$.

Suppose $\xi\epsilon[X, H]$ and $\xi\notin(X, H^*)$. By the fact $H\cong H^*$, each of two groups H and $X/(X, H^*)$ is the character group of the other. Now construct an orthogonal pair H and $X/(X, H^*)$. Then $x\xi=x\xi^*\neq 0$ (where $\xi^*=\xi+(X, H^*)$) for some $x\epsilon H$, which is a contradiction. Then we have $[X, H]\subset(X, H^*)$. Similarly, we have $(X, H^*) \subset[X, H]$. Hence we conclude $[X,H]=(X, H^*)$. In the same way, we see that $[X, H_i]=(X,H_i{}^*)$.

Replace H by H^* and put $X/[X,H]=X/(X,H^*)$ and $X/[X,H_i]=X/(X,H_i^*)$ in the first paragraph of proof of this theorem. Then we obtain $H^*\cong\Sigma H_i^* g_i$, and H^* is open. Then the set $\Sigma H_i^* g_i$ is closed in G^*, which is (P. 1). The character group $X/(X,H_2^*)$ of H_2^* implies $X/[X,H_2]=X/(X,H_2^*)\cong K\cdot D^p$, which shows (P. 2).

The subgroup (X, H^*) is the character group of the discrete group G^*/H^*, and so $[X,H]=(X,H^*)$ is compact, which is (P. 3). Suppose that $x^*\neq y^*$ (where $x^*, y^*\in G/H$) implies $R_x^*=R_y^*$ Then we have $\xi x^*=\xi y^*$ for all $\xi\in[X,H]$, and so $x^*=y^*$. This is a contradiction. Therfore the algebraic mapping: $x^*\to R_x^*$ is injective, which shows (P. 4). The proof of THEOREM is finished.

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