◇論　　文◇

ON THE COMPACTNESS OF THE STRUCTURE SPACE OF A RING

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1. INTRODUCTION. Jacobson [1] has shown that a topology may be defined on the set $S(A)$ of primitive ideals of any non-radical ring A. With this topology $S(A)$ is called the STRUCTURE SPACE of A. The topology is given by defining closure: If $T=\{P\}$ is a set of primitive ideals then T is the set of primitive ideals which contain

$$D_T=\bigcap\{P\mid P\in T\}.$$

It is well known that if A has an identity element, then $S(A)$ is compact [2, pp. 208]. Moreover M. Schreiber [3] has recently observed that if every two-sided ideal of A is finitely generated, then $S(A)$ is again compact. However, since the condition that A has an identity element neither implies nor is implied by the condition that every ideal of A be finitely generated, it is clear that neither of these conditions is necessary in order that $S(A)$ be compact. R. L. Blair and L.C. Eggan [4] investigated this situation throughly and found a condition that is both necessary and sufficient for the compactness of $S(A)$ as a consequence of a general lattice-theoretic result. They also obtained a remarkable result for a class of rings consisting of those rings A such that no non-zero homomorphic image of A is a radical ring, stating that the structure space of such ring is compact if and only if A is generated, as an ideal, by a finite number of elements. Recently a direct proof was given by Taikyun Kwun [5] using open sets instead of closure. In view of the fact that the Jacobson radical of an arbitrary ring plays an important role in the structure theory of rings, the author tried to find a relationship between the notion of radical and the compactness of a structure space. As a result he found that for a certain class of rings the modularity of the radical is both necessary and sufficient for compactness of $S(A)$, and for another class of rings the condition is that A is generated as an ideal by a finite number of elements. Section 2 is entirely due to M. Schreiber [4] and the author has found his method very useful in searching for a link between the compactness and the radical. Section 3 is concerned with properties of the modular ideal which will be used in the proofs of the main theorems. Section 4 and 5 contain the main theorems.

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2. OPEN BASIS OF THE TOPOLOGY. For each $x\in A$ write (x) for the principal two-sided ideal generated by x, and let

$$U_x=\{P\mid P\not\supseteq(x), \text{ for all } P\in S(A)\}.$$

PROPOSITION 1. $\{U_x\}_{x\in A}$ is an open basis of the topology.

PROOF. Since the set $\{P\mid P\supseteq(x)\}$ is clearly closed, its complement U_x is open. Let U be an open subset of $S(A)$, and let $F=U^c$ (the complement of U). Now suppose $P\in U$. Since $F=\bar{F}=\{P'\mid P'\supseteq D_F\}$ and $P\notin F$, we have P D_F. Hence there exists an element a in A such that a belongs to D_F but not to P, so that P (a), that is, $P\in U_a$. Suppose $P'\in U_a$, then $P'\not\supseteq(a)$ and $P\not\supseteq D_F$, so that $P'\notin F$, or $P'\in U$. Hence $P\in U_a\subseteq U$.

PROPOSITION 2. If a ring A has an identity then $S(A)$ is compact.

PROOF. We prove that any basic open cover has a finite subcover. By Proposition 1, the collection

$$\{U_x\}_{x \in A}$$

is an open cover of S(A). Then

$$S(A) = \bigcup \{U_x\}_{x \in A}$$
$$= \{P \mid \exists\ \nu \text{ such that } P \not\supseteq (a_\nu)\}$$
$$= \{P \mid I \not\supseteq \Sigma_{a \in A}\ (a)\}.$$

Write

$$I = \Sigma_{a \in A}\ (a).$$

In a ring with an identity every two-sided ideal can be imbedded in a primitive ideal [6]. But

$$\{P \mid P \not\supseteq I\} = S(A)$$

exhausts all primitive ideals. Hence I=A, so that the identity element 1 is in I. Hence there exist $b_1, \cdots, b_n$ in $(a)+\cdots+(b)$ such the identity $1 = b_1 + \cdots + b_n$, so that

$$A = (a) + \cdots + (b) = I.$$

But this means that there exists a finite subset $E = \{a, \cdots, b\}$ of A such that

$$S(A) = \{P \mid P \not\supseteq \Sigma_{a \in A}(a)\}$$
$$= \bigcup \{U_a\}_{a \in E}.$$

This proves the Proposition.

3. MODULAR IDEALS.

DEFINITION. A two-sided ideal P is called modular if and only if there exists an element e in A such that for all a of A, a-ea, a-ae $\in$ P. The element e is called an identity modulo P.

Evidently, if a two-sided ideal P of ring A is modular with an identity e modulo P, then A/P is a ring with an identity e+P.

PROPOSITION 3. An intersection of finite number of modular two-sided ideals is modular.

PROOF. Let P and P′ be modular two-sided ideals of a ring A, and let e_1 and e_2 be the identities modulo P and P′ respectively. Then it suffices to show that the intersection of P and P′ is modular. Now put

$$e = e_2 \circ e_1 = e_2 + e_1 - e_2 e_1$$

and

$$A(1-e) = \{x - xe \mid x \in A\},$$
$$A(1-e_i) = \{x - xe_i \mid x \in A\}.$$

It follows $A(1-e) = A(1-e_2)(1-e_1)$

since
$$x - xe = x - x(e_2 \circ e_1)$$
$$= x - x(e_2 + e_1 - e_2 e_1)$$
$$= x - xe_2 - xe_1 + xe_2 e_1$$
$$= (x - xe_2) - (x - xe_2)e_1.$$

On the other hand, $A(1-e_2) \subseteq A$, and this implies

$$A(1-e_2)(1-e_1) \subseteq A(1-e_1).$$

But $A(1-e_1) \subseteq P$. Hence we have $A(1-e) \subseteq P$. Since $A(1-e_2) \subseteq P'$, we have $A(1-e_2)(1-e_1) \subseteq P'(1-e_1)$. Therefore

$$A(1-e) \subseteq P' \text{ and } A(1-e) \subseteq P \cap P'.$$

A similar argument implies

$$(1-e)A \subseteq P \cap P'.$$

Hence $P \cap P'$ is modular.

PROPOSITION 4. Every two-sided modular ideal can be imbedded in a primitive ideal.

PROOF. Let (I:A) denote the set of element a of a ring A such that $Aa \subseteq I$ where I is a modular maximal right ideal. Since

$$(I:A) = (O:M)$$

where M is an irreducible A-module [2, Proposition 2, pp. 6], it follows that (I:A) is primitive ideal. Now let B be a modular two-sided ideal. Since it is well known that every modular right ideal can be imbedded in a modular maximal right ideal, regarding B as a modular right ideal, we can put $B \subseteq I$. But $Ab \subseteq B$ for every element b of B. Hence $B \subseteq (I:A)$.

4. COMPACTNESS AND THE MODULARITY OF THE RADICAL.

By Proposition 5 below, the modularity of the radical of a ring A is sufficient for the compactness of S(A). But it is not necessary for the same reason stated in the INTRODUCTION. An effort was done in searching a class of rings for which the compactness of S(A) necessitate the modularity of the radicals.

PROPOSITION 5. If the radical R of a ring A is modular, then S(A) is compact.

PROOF: Let e be the identity modulo R. Then A/R is clearly a ring with an identity. Since A/R has an identity, the structure space S(A/R) is compact. Now consider a primitive ideal P. Then we have

$$(A/R)/(P/R) \cong A/P.$$

Hence P/R is a primitive ideal in A/R for A/P is primitive ring. Conversely any primitive ideal in A/R is of the form P/R, where P is a primitive ideal. Since the correspondence

$$P \longrightarrow P/R$$

preserves arbitrary intersection, it follows that it is a homeomorpism of S(A) onto the structure space S(A/R) of A/R. Therefore S(A) is compact.

Now we consider a class of ring with the property that

(C) For every U_a of $\{U_a\}_{a \epsilon A}$, DU_a is modular.

For such a ring A, a result can be obtained as follows:

PROPOSITION 6. For a ring A satisfying the condition (C), if S(A) is compact, then the radical R of A is modular.

PROOF: It suffices to show that any basic open cover has a finite subcover. Since $\{U_x\}_{x \epsilon A}$ is an open cover, there exists a subcover $\{U_a\}_{a \epsilon E}$ where E is a finite subset of A. Then

$D \cup \{U_a\} = D_{S(A)} = R$. (R: the radical of A)

since

$$\{U_a\}_{a \epsilon E} = S(A).$$

But

$$D \cup \{U_a\} \supseteq DU_a \cap DU_b \cap \cdots \cap DU_c,$$

where $a, b, \cdots, c \epsilon E$.

Hence

$$R \supseteq DU_a \cap DU_b \cap \cdots \cap DU_c,$$

where $a, b, \cdots, c \epsilon E$.

By hypothesis, each $\{DU_a\}$, $a \epsilon E$, is modular. Then it follows that the radical R of A is modular by Proposition 3. This proves the Proposition.

Combining proposition 5 and 6, we obtain the following:

THEOREM 1. Let A be a ring with a property that each DU_a is modular. Then the structure space of A is compact if and only if the radical R of A is modular.

5. COMPACTNESS AND MODULARITY OF PRINCIPAL TWO-SIDED IDEALS. It is pointed out explicitly in [5] that the condition that A is generated, as an ideal, by a finite number of elements is sufficient for the comactness of S(A) regardless of the type of A. We consider a class of rings with the property that

(C′) Every principal two-sided ideal is modular.

PROPOSITION 7. For a ring A satisfying the condition (C′), if S(A) is compact, then the ring A is generated, as an ideal, by a finite number of elements.

PROOF: Consider the basic open cover $\{U_x\}_{x \epsilon A}$ Then we have a finite subcover

$$\{U_a\}_{a \epsilon E}$$

where E is a finite subset of A. and

$$S(A) = \bigcup \{U_a\}_{a \epsilon E}$$
$$= \{P \mid P \not\supseteq \Sigma_{a \epsilon E}(a)\}.$$

Write

$$B = \Sigma_{a \epsilon E}(a).$$

Since B contains a principal ideal, it is modular. Now suppose B is proper ideal of A. By proposition 4, B can be imbedded in a primitve ideal. This is a contradiction to the fact that

$$\{P \mid P \not\supseteq B\} = S(A)$$

exhausts all primitive ideals. Hence B=A. We state this fact in the following form:

THEOREM 2. Let A be a ring with the property that every principal two-sided ideal is modular. Then the structure space of A is compact if and only if the ring A is generated, as an ideal, by a finite number of elements.

REMARKS: Consider a ring A with a property that

(C″) No non-zero homomorphic image of A is a radical ring.

R.L. Blair and L.C. Eggan have proved that, for such a ring, the structure space is compact if and only if A is generated as an ideal by a finite number of elements. It would be interesting to clarify the relations among the classes of rings satisfying codition (C), (C′) and (C″).

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研究問題의 紹介

1. 1962年 9月 24日 美國數學會 Bulletin에 提出되고, Bull Vol 69, No 1, P 41에 紹介된 問題39番, 出題者 Joseph Hammer

n^2 개의 數가 주어 졌을때(모두 다를 必要는 없음), 이들을 어떻게 配列하면, 그 配列한 結果로 얻는 行列式의 값이 미리 定해 놓은 값과 같아 지는가?

2. 1963年 8月 7日 Bull에 提出되고, Vol 69, No 6, P 738에 紹介된 o. Taussky 의 問題, 10番.

A, B 를 代數的閉體위의 n×n 行列이라고 하자. A 와 可換인 任意의 行列X와 B가 可換이면, B는 그 體에 係數를 가지는 A에 關한 多項式이라야 한다. (H.M. Wedderburn의 Lectures on matrices, Amer. Math. Soc. Colloq. Publ. Vol 17, 1934를 보라)

最近에 M. Marcus와 N.A. Khan 은 (Canad. J. Math. 12(1960), 259—277) 同結果를 A가 AX−XA와 可換이면 항상 X는 XB−BX와 可換이라는 假定下에 證明하였다.

더욱 最近에는 M.F. Smiley (Canad. J. Math. 13 (1961), 353—355)에 의하여 이것이 Characteristic이 0 혹은 적어도 n 인 경우를 許容하고 任意位數의 commutativity 일때에 까지 一般化되었다.

그러면 Smiley 의 이 結果는 代數的閉體아닌 體에 대하여도 眞인가?

3. 自然數에 있어서 1부터 2n 까지의 數를 한번씩 써서 n개의 짝

$(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n), \ a_i < b_i$

을 만든다. 지금

$c_i = a_i + b_i, \ d_i = b_i - a_i$

라 놓아서 얻는 2n 개의 數 c_i, d_i 는 서로 다를수 있겠는가? 이 問題는 Mok-Kong Shen 과 Tsen-Pao Shen 에 依하여, 提起된 것으로서, 1962年 9月 6日美國數學會의 Bulletin 에 提出되어, Bull Vol 68, No 6, p 557에 問題39番으로서 紹介되었던 것이다.

n=1, 2일때는 答은 No 임을 簡單히 알수 있다. 따라서 問題가 되는 것은 n≧3일때이다. 그들에 의한 例로서

n=3 : (1, 5), (2, 3), (4, 6)
n=6 : (1, 10), (2, 6), (3, 9), (4, 11), (5, 8), (7, 12)
n=8 : (1, 10), (2, 14), (3, 16), (4, 11), (5, 9), (6, 12), (7, 15), (8, 13),

을 들수 있다.

그런데 1963年 1月6日 Bull 에 提出되고, Vol 69, No 3, p 333에 紹介된 M. Slater 의 問題1番은 이 問題에 대하여 다음과 같이 主張하였다. 즉

1부터 n 까지의 數와 n+1부터 2n 까지의 數로 짝을 만들때 2n개의 和와 差 $b_i \pm i$ 가 모두 다르게 할수 있겠는가?

Slater 에 의하면 n=2, 3, 6 일때는 不可能하고, 그외의 n에 대하여는 可能하다고 豫想이 되어 있다. 그에 의하면 n×n 型 Chess 에서 n 개의 queen 을 느러 놓고 queen 끼리는 서로 攻擊하지 않는다는 問題와 關聯이 있다고 한다. 그리고 Slater 는 다음과 같은 例를 들고 있다.

n=4 : (1, 7), (2, 5), (3, 8), (4, 6),
n=7 : (1, 9), (2, 14), (3, 12), (4, 10) (5, 8), (6, 13), (7, 11),
n=9 : (1, 14), (2, 18), (3, 11), (4, 13), (5, 16), (6, 12), (7, 17), (8, 15), (9, 10)