PROJECTIVE MOTIONS IN NON-RIEMANNIAN K*-SPACES I

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§1. Introduction.

We consider an N-dimensional analytic space A_N with a symmetric connection Γ^i_{jk}. A_N is called an AK^*_N-space if its curvature tensor

$$B^i_{jkl}=\Gamma^i_{jl,k}-\Gamma^i_{jk,l}+\Gamma^h_{jl}\Gamma^i_{hk}-\Gamma^h_{jk}\Gamma^i_{hl}$$

satisfies the relation

$$(1.1)\qquad B^i_{jkl;m}=B^i_{jkl}K_m ,$$

where a comma means ordinary partial differentiation with respect to coordinates, a semi-colon denotes covariant differentiation with respect to Γ^i_{jk} and K_m a non-zero vector. In this paper, the author will discuss the projective motion of torse-forming form, in AK^*_N-space, defined by

$$(1.2)\qquad \bar{x}^i=x^i+v^i(x)dt, \qquad v^i_{;j}=\rho(x)\delta^i_j+\phi_j(x)v^i(x)$$

where $\rho(x)$ means any function of x and ϕ_j denotes a certain covariant vector.

In what follows, we assume the existence of projective motion of the torse-forming form (1.2), that is, we assume the condition

$$(1.3)\qquad \pounds\Gamma^i_{jk}=\delta^i_j\Psi_k+\delta^i_k\Psi_j , \quad \text{or} \quad v^i_{;j;k}=B^i_{jkl}v^l+\delta^i_j\Psi_k+\delta^i_k\Psi_j$$

which is the necessary and sufficient condition that the vector $v^i(x)$ defines a projective motion, where the symbol $\pounds$ means the Lie derivative with respect to $v^i(x)$ and $\Psi_j(x)$ is a certain non-zero vector.

We introduce a quantity Π^o_{jk} by

$$(1.4)\qquad \Pi^o_{jk}=-\frac{1}{N^2-1}(NB_{jk}+B_{kj}), \qquad B_{jk}=B^h_{jkh} .$$

Then we can obtain Wyle's projective curvature tensor P^i_{jkl} of the form:

$$(1.5)\qquad P^i_{jkl}=B^i_{jkl}+\Omega^i_{jkl} ,$$

where we have put

$$\Omega^i_{jkl}=\delta^i_l\Pi^o_{jk}-\delta^i_k\Pi^o_{jl}-\delta^i_j\Pi^o_{kl}+\delta^i_j\Pi^o_{lk}\ . \tag{1. 6}$$

If we introduce an other quantity P^o_{jkl} by

$$P^o_{jkl}=\Pi^o_{jk;l}-\Pi^o_{jl;k}\ , \tag{1. 7}$$

the well known equation $\pounds B^i_{jkl}=(\pounds\Gamma^i_{jk})_{;l}-(\pounds\Gamma^i_{jl})_{;k}$ gives

$$\pounds\Pi^o_{jk}=v^m\Pi^o_{jk;m}+v^m_{;k}\Pi^o_{jm}+v^m_{;j}\Pi^o_{mk}=\Psi_{j;k}\ . \tag{1. 8}$$

It is known [1] that the integrability condition of (1. 3) is

$$\text{(i)}\quad \pounds P^i_{jkl}=\pounds B^i_{jkl}+\delta^i_l\Psi_{j;k}-\delta^i_k\Psi_{j;l}-\delta^i_j\Psi_{k;l}+\delta^i_j\Psi_{l;k}=0\ , \tag{1. 9}$$

$$\text{(ii)}\quad \pounds P^o_{jkl}=-\Psi_hP^h_{jkl}\ .$$

§2. Projective motion and two cases.

In order to discuss the projective motion of torse-forming form, we recall the following lemma which has proved by K. Takano [2] :

LEMMA. *If an AK^*_N-space* ($N\geqq 3$) *admits an infinitesimal projective motion, the motion should be of the form:*

$$\bar{x}^i=x^i+v^i(x)dt,\qquad \pounds\Gamma^i_{jk}=\delta^i_j\Psi_k+\delta^i_k\Psi_j,\quad \Psi_k=\frac{1}{N-2}\pounds K_k\ .$$

It follows from the above Lemma that the equation (1. 3) may be rewritten in the form:

$$v^i_{;j;k}=B^i_{jkl}v^l+\frac{1}{N-2}(\delta^i_j\,[v^aK_{k;a}+\rho K_k+v^aK_a\phi_k\,]+\delta^i_k\,[v^aK_{j;a}+\rho K_j+v^aK_a\phi_j]\,)\ . \tag{2. 1}$$

Differentiating (1. 2) covariantly, we obtain

$$v^i_{;j;k}=\phi_{j;k}v^i+\phi_j\phi_kv^i+\rho_k\delta^i_j+\rho\phi_j\delta^i_k\ .$$

Substituting this equation into (2, 1), we have

$$(2.\ 2)\qquad B^i_{jkl}v^l=\phi_{j:k}v^i+\phi_j\phi_k v^i+\rho_k\delta^i_j+\rho\phi_j\delta^i_k$$
$$-\frac{1}{N-2}(\delta^i_j[v^aK_{k:a}+\rho K_k+v^aK_a\phi_k]+\delta^i_k[v^aK_{j:a}$$
$$+\rho K_j+v^aK_a\phi_j]) .$$

The identity $B^i_{jkl}v^kv^l=0$ gives the following equation by means of (2, 2):

$$(2.\ 3)\qquad \phi_{j:k}v^iv^k+\phi_j\phi_k v^iv^k+\rho_k v^k\delta^i_j+\rho\phi_j v^i$$
$$-\frac{1}{N-2}(\delta^i_j[K_{k:a}v^kv^a+\rho K_k v^k+v^aK_a\phi_k v^k]$$
$$+v^i[v^aK_{j:a}+\rho K_j+v^aK_a\phi_j])=0 .$$

Contraction of $i=j$ gives

$$(2.\ 3)_1\qquad \phi_{j:k}v^jv^k+\phi_j\phi_k v^jv^k+N\rho_j v^j+\rho\phi_j v^j$$
$$=\frac{N+1}{N-2}(K_{j:a}v^jv^a+\rho K_j v^j+v^aK_a\phi_j v^j) .$$

Multiplying v^j to the equation (2, 3) and summing over j, we get the following equation for non-zero vector v^i:

$$(2.\ 3)_2\qquad \phi_{j:k}v^jv^k+\phi_j\phi_k v^jv^k+\rho_j v^j+\rho\phi_j v^j$$
$$=\frac{2}{N-2}(K_{j:a}v^jv^a+\rho K_j v^j+v^aK_a\phi_j v^j) .$$

Comparing these two equations, we have

$$(2.\ 4)\qquad (N-2)\rho_j v^j=K_{j:a}v^jv^a+\rho K_j v^j+v^aK_a\phi_j v^j .$$

Substitution of (2, 4) into (2, 3) gives, for non-zero v^i.

$$(2.\ 5)\qquad \phi_{j:k}v^k=\frac{1}{N-2}(v^kK_{j:k}+\rho K_j+v^kK_k\phi_j)-\phi_j\phi_k v^k-\rho\phi_j .$$

Multiplying v^j to the equation (2, 5) and summing over j, we have

$$(2.\ 5)'\qquad \phi_{j:k}v^jv^k=\rho_j v^j-\rho\phi_j v^j-\phi_j v^j\phi_k v^k .$$

Differentiating (2.4) convariantly, we have

$$(N-2)\rho_{a;j}v^a+(N-2)(\rho\rho_j+\rho_a v^a\phi_j)=K_{a;b;j}v^a v^b$$
$$+\rho K_{j;b}v^b+K_{a;b}v^a v^b\phi_j+\rho K_{a;j}v^a+K_{a;b}v^a v^b\phi_j+K_a v^a\rho_j$$
$$+\rho K_{a;j}v^a+\rho^2 K_j+\rho K_a v^a\phi_j+\rho K_j\phi_a v^a+v^a K_a v^b\phi_b\phi_j$$
$$+K_{a;j}v^a\phi_b v^b+v^a K_a\phi_{b;j}v^b+\rho K_a v^a\phi_j+v^a K_a v^b\phi_b\phi_j\ .$$

Multiplying v^j and making use of (2.5)′ and (2.4), we obtain

$$(2.\ 6)\qquad (N-2)\rho_{a;b}v^a v^b-K_{a;b;c}v^a v^b v^c$$
$$=2K_{a;b}v^a v^b\phi_c v^c+2\rho K_{a;b}v^a v^b+2\rho_a v^a K_b v^b.$$

Now, we are going to classify the projective motion by using above results. Contraction of $i=k$ in the equation (2.2) gives

$$B^h_{jhl}v^l=\phi_{j;h}v^h+\phi_h v^h\phi_j+\rho_j+N\rho\phi_j-\frac{N+1}{N-2}(v^a K_{j;a}+\rho K_j+v^a K_a\phi_j).$$

Comparing this equation with (2.5), we get

$$(2.\ 7)\qquad B^h_{jhl}v^l=\rho_j+(N-1)\rho\phi_j-\frac{N}{N-2}(v^a K_{j;a}+\rho K_j+v^a K_a\phi_j).$$

Differentiating (2.7) covariantly, we have

$$(K_m+\phi_m)B^h_{jhl}v^l+\rho B^h_{jhm}=\rho_{j;m}+(N-1)\rho_m\phi_j+(N-1)\rho\phi_{j;m}$$
$$-\frac{N}{N-2}(\rho K_{j;m}+\phi_m K_{j;a}v^a+v^a K_{j;a;m}+\rho_m K_j+\rho K_{j;m}$$
$$+\rho K_m\phi_j+v^a K_a\phi_m\phi_j+v^a K_{a;m}\phi_j+v^a K_a\phi_{j;m}).$$

Multiplying both hand sides of the above by $v^j v^m$ and summing over j and m, it follows from (2.5)′ and (2.6) that

$$(2.\ 8)\qquad \rho_{j;m}v^j v^m+(K_m v^m+2\phi_m v^m+2\rho)(\rho\phi_j v^j-\rho_j v^j)=0\ .$$

On the other hand, by making use of (2.4) and (2.7), we have

$$-B_{jl}v^j v^l=(N-1)(\rho\phi_j v^j-\rho_j v^j)\ .$$

By the general rule of Lie differentiation and (1.2), we get

$$\pounds B_{jk}=B_{jk}v^aK_a+\rho B_{jk}+v^aB_{ak}\phi_j+\rho B_{jk}+v^aB_{ja}\phi_k .$$

Comparing these two equations, we have

$$-(\pounds B_{jk})v^jv^k=(N-1)(v^aK_a+2\phi_av^a+2\rho)(\rho\phi_jv^j-\rho_jv^j).$$

Substitution of this equation into (1.4) gives

$$(\pounds\Pi^0{}_{jk})v^jv^k=(v^aK_a+2\phi_av^a+2\rho)(\rho\phi_jv^j-\rho_jv^j). \tag{2.9}$$

By using the Lemma of §1, we have

$$\Psi_{j;k}=\frac{1}{N-2}(\rho K_{j;k}+K_{j;a}v^a\phi_k+v^aK_{j;a;k}+\rho_kK_j+\rho K_{j;k}+\rho K_k\phi_j$$
$$+v^aK_a\phi_j\phi_k+v^aK_{a;k}\phi_j+v^aK_a\phi_{j;k}).$$

Multiplying both hand sides of the above by v^jv^k, and summing over j and k, it follows from (2.5)′ and (2.6) that

$$\phi_{j;k}v^jv^k=\rho_{j;k}v^jv^k. \tag{2.10}$$

Hence, we have the following equation by combining the equations (1.8), (2.9), (2.10) and (2.8):

$$(K_mv^m+2\phi_mv^m+2\rho)(\rho\phi_jv^j-\rho_jv^j)=0 . \tag{2.11}$$

Therefore we have proved

THEOREM 1. *If a general* AK^*_N*-space admits a projective motion of torse-forming form* (1.2) ($N\geqq 3$), *then there exist two cases*:

$$\text{(i)}\ \rho\phi_jv^j-\rho_jv^j=0 , \quad \text{(ii)}\ K_jv^j+2\phi_jv^j+2\rho=0 .$$

Differentiating $\rho\phi_jv^j-\rho_jv^j=0$ covariantly, we obtain

$$\rho_m\phi_jv^j+\rho\phi_{j;m}v^j+\rho^2\phi_m=\rho_{j;m}v^j+\rho\rho_m , \tag{2.12}$$

in which we have used $\rho\phi_jv^j-\rho_jv^j=0$.

Multiplying both sides of (2.12) by v^m and making use of (2.5)′ and

$\rho\phi_j v^j - \rho_j v^j = 0$ give

$$\rho^2 \phi_j v^j = 0 \tag{2.13}$$

Hence, we have

THEOREM 2. *The first case of theorem* 1 *is degenerated into the following two parts again*:

$$\text{(i)}\ \rho = 0\ , \qquad \text{(ii)}\ \phi_j v^j = 0\ .$$

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