ON A CONTINUOUS MAPPING BETWEEN PARTIALLY ORDERED SETS WITH SOME TOPOLOGY

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1. Introduction and Notations

Let P be a partially ordered set. By the interval topology of P, we mean that defined by taking the closed intervals $[a, b]$, $[-\infty, a]$, and $[a, \infty]$ of P as a sub-base of closed sets. Let f be a mapping of a partially ordered set P_1 into another partially ordered set P_2. In this paper, we first obtain a necessary and sufficient condition that f be a continuous in their interval topologies. This condition, stated in theorem 1, can be applied to show that if f is a complete isotone of a complete lattice into a complete lattice, then f is a continuous in their interval topologies.

N. Funayama [2] has introduced an *imbedding operator* ϕ in the family of subsets of P, and has defined a *completion* P_ϕ of P by the imbedding operator ϕ. And he has obtained a lot of interesting results that P is imbedded into some complete lattice. In theorem 2 we consider conditions under which P is continuously imbedded into a complete lattice with respect to their interval topologies.

T. Naito [3] has introduced the concept of CP-ideal topology. In §3, we shall deal with similar results of §2 with respect to CP-ideal topology.

We shall use I, I_α, $I_{\alpha\beta}$, J, J_α, $J_{\alpha\beta}$, to denote closed intervals in § 2 and to denote CP-ideals or dual CP-ideals in § 3. We denote the join and the meet of two elements x and y of a lattice by $x \cup y$ and $x \cap y$ respectively, the join and the meet of all elements of a set $\{a_\alpha | \alpha \in \Delta\}$ by $\sup_{\alpha \in \Delta} a_\alpha$ and $\inf_{\alpha \in \Delta} a_\alpha$ respectively. $A \vee B$ and $\bigvee_{\alpha \in \Delta} X_\alpha$ will be used to denote the set union of two sets A and B, and of sets of the family $\{X_\alpha | \alpha \in \Delta\}$, and $A \wedge B$ and $\bigwedge_{\alpha \in \Delta} X_\alpha$ are the set intersections of them. Finally, the complement of a set A will be denoted by A^c.

2. Interval topology.

We here note that if a subset S of P is a closed set in it's interval topology, then S may be expressed as an intersection of the sets which are unions of a finite number of closed intervals in P:

$$\dot{S} = \bigwedge_{\alpha}\{\bigvee_{\beta=1}^{n\alpha} I_{\alpha\beta}\}$$

where $I_{\alpha\beta}$ is the form of $[a,b]$, $[a,+\infty]$, or $[-\infty,b]$. Thus an open subset O in P is expressed as

$$O=\bigvee_{\alpha}\{\bigwedge_{\beta=1}^{n\alpha} I^c_{\alpha\beta}\}$$

Let P be a partially ordered set. A subset S of P is called to be *covered* by a finite closed intervals of P if there exist a finite number of closed intervals I_n such that $S \subseteq \bigvee_n I_n$.

We first prove the following theorem:

THEOREM 1. *Let P_1 and P_2 be two partially ordered sets, and f a mapping of P_1 into P_2. f is continuous in their interval topologies if and only if for any closed interval J of P_2 and any element x of P_1 such that $x \notin f^{-1}(J)$, there exists a covering of $f^{-1}(J)$ by means of a finite number of closed intervals none of which contains x.*

PROOF. Suppose that f is a continuous mapping of P_1 into P_2. And $x \notin f^{-1}(J)$ for a closed interval J of P_2 and an element x of P_1. Since $f^{-1}(J)$ is a closed set in P_1, it may be expressed as following $f^{-1}(J)=\bigwedge_{\alpha}\{\bigvee_{\beta=1}^{n\alpha} I_{\alpha\beta}\}$, where $I_{\alpha\beta}$ is a closed interval in P_1. Thus $x \notin \bigvee_{\beta=1}^{n\alpha_0} I_{\alpha_0\beta}$ for some α_0. Moreover $f^{-1}(J) \subseteq \bigvee_{\beta=1}^{n\alpha_0} I_{\alpha_0\beta}$ and $x \notin I_{\alpha_0\beta}$ $(1 \leqq \beta \leqq n_{\alpha_0})$. Conversely, for an element x of P_1, let O_2 be a neighborhood of $f(x)$ in P_2. It suffices to show that for some open subset O_1 contining x, $O_1 \subseteq f^{-1}(O_2)$. Thus we may assume that O_2 is an open set in P_2, which may be expressed as $O_2=\bigvee_{\alpha}\{\bigwedge_{\beta=1}^{n\alpha} J^c_{\alpha\beta}\}$, where $J_{\alpha\beta}$ is a closed interval or the empty set or P_2. And there exists a closed intervals $J_{\alpha_0\beta}$ such that $f(x) \notin J_{\alpha_0\beta}$, i.e. $x \notin f^{-1}(J_{\alpha_0\beta})$ for some α_0 and all β corresponding to α_0. By the hypotheses, there are a finite number of closed intervals I^β_n $(\not\ni x)$ such that $f^{-1}(J_{\alpha_0\beta}) \subseteq \bigvee_n I^\beta_n$, i.e. $(\bigvee_n I^\beta_n)^c \subseteq f^{-1}(J^c_{\alpha_0\beta})$ for each β. On the other hand, $x \in \bigwedge_{\beta=1}^{n\alpha_0} (\bigvee_n I^\beta_n)^c \subseteq \bigwedge_{\beta=1}^{n\alpha_0} f^{-1}(J^c_{\alpha_0\beta}) \subseteq f^{-1}(O_2)$, which completes the proof.

A mapping f of a partially ordered set P_1 into P_2 is called a *complete isotone* if $\sup_{\alpha\in\Delta} x_\alpha$, $\sup_{\alpha\in\Delta} f(x_\alpha)$ exist and $x=\sup_{\alpha\in\Delta} x_\alpha$ implies $f(x)=\sup_{\alpha\in\Delta} f(x_\alpha)$, and it's dual. Theorem 1 can be applied to show the following

COROLLARLY 1. *Let f be a complete isotone of a complete lattice P_1 into a complete lattice P_2. Then f is continuous in their interval topologies.*

PROOF. Let J be a closed interval in P_2 and x an element in P_1 such that $x\notin f^{-1}(J)$. We shall show that there is a closed interval I in P_1 not containing x such that $f^{-1}(J)\subseteq I$. If the set $S=\{y\in P_1 | f(y)\in J\}$ is empty, then we may take the empty set as I. Therefore we may assume that S is non-empty. Let $a=\inf S$, $b=\sup S$. If we suppose $x\in[a,b]$, then $f(x)\in J$ because f is a complete isotone. It follows that $x\in f^{-1}(J)$ which is contrary. Clearly we see that $f^{-1}(J)\subseteq[a,b]$, which completes the proof.

N. Funayama [2] has defined an imbedding operator ϕ in the family of subsets of a partially ordered set P. A is called ϕ-closed if $\phi(A)=A$. All the ϕ-closed sets form a complete lattice P_ϕ under set inclusion. P_ϕ is called the *completion* of P by the imbedding operator ϕ. And he has proved that if a collection $\Omega=\{A_\lambda\}$ of subsets of P satisfies the following conditions: (i) every A_λ is an ideal of P, i.e. $a\in A_\lambda$ and $x\leqq a$ then $x\in A_\lambda$, (ii) every principal ideal is a member of Ω, (iii) Ω is M-complete, i.e. for any subset $\{B_\lambda\}$ of Ω, $\wedge_\lambda B_\lambda\in\Omega$, (iv) $P\in\Omega$, then there exists an uniquely determined imbedding opertor ϕ on P such that $\Omega=P_\phi$.

The theorem 2 of [2] says that let ϕ be an imbedding operator on P, then P is imbedded into P_ϕ by f ; $f(a)=(a]$ (=principal ideal generated by a), where f is O-isomorphism, i.e. $f(a)\geqq f(b)$ if and only if $a\geqq b$.

The lemma 2 and theorem 2 of [2] and theorem 1 give us the following lemma:

LEMMA 1. *Let P be a partially ordered set. If there is a collection Ω satisfying (i)$\sim$(iv) in P, and if the mapping f: $f(a)=(a]$ of P into Ω satisfies the hypothesis of theorem* 1, *then P is continuousely imbedded into the complete lattice P_ϕ $(=\Omega)$.*

Hence, by above lemma 1 and corollarly, we have

THEOREM 2. *Under the hypotheses of lemma* 1, *if g is a complete isotone* *of* P_ϕ *into a complete lattice L, then P is continuousely imbcdded in L by* $g\circ f$ *in to L in their interval topologies.*

3. CP-ideal topology

In this section, we denote P to be a lattice. An ideal I is said to be a *prime* ideal if and only if $x\cap y$ implies $x\in I$ or $y\in I$. A prime ideal I is called a *CP-ideal* if and only if the following condition holds: if $\{x_\alpha | \alpha\in\Delta\}\subseteq I$ and there exists $\sup_{\alpha\in\Delta} x_\alpha$, then $\sup_{\alpha\in\Delta} x_\alpha \in I$. Dually, a dual prime ideal and a dual CP-ideal are defined (T. Naito [3]). The union of {all CP-ideals of P}, {all dual CP-ideals of P} and $\{\phi, P\}$ is denoted by $\mathfrak{L}\mathfrak{P}$, where ϕ is the empty set. We recall that the CP-ideal topology of a lattice P is that defined by taking the elements of $\mathfrak{L}\mathfrak{P}$ as a sub-base of closed sets of the space P.

In the same way as in §2, We can prove the following

THEOREM 3. *Let* P_1 *and* P_2 *be two lattices, and f is mapping of* P_1 *into* P_2. *f is continuous in their CP-ideal topologies if and only if for any member J of* $\mathfrak{L}\mathfrak{P}$ *of* P_2 *and any element x of* P_1 *such that* $x\notin f^{-1}(J)$ *there exists a covering of* $f^{-1}(J)$ *by means of a finite number of members of* $\mathfrak{L}\mathfrak{P}$ *none of which contains x.*

As a corollarly of the theorem 3, we also have

COROLLARLY 2. *Let f be a complete isotone of a complete lattice* P_1 *into a complete lattice* P_2. *Then f is a continuous mapping of* P_1 *into* P_2 *in their CP-ideal topologies.*

PROOF. Let J be a member of $\mathfrak{L}\mathfrak{P}$ of P_2 and x an element in P_1 such that $x\notin f^{-1}(J)$. We shall show that there exists a member I of $\mathfrak{L}\mathfrak{P}$ of P_1 not containing x such that $f^{-1}(J)\subseteq I$. We consider J into three cases:

(i) J is a nonvoid CP-ideal. Let $S=\{y_r\in P_1 | f(y_r)\in J\}$. If $S=\phi$, i.e. $f^{-1}(J)=\phi$ we then take the empty set as I. And we may assume $S\neq\phi$. Set $a=\sup S$. Then $(a]$ is a CP-ideal of P_1. For, if $u\cap v\in(a]$, then $f(u)\cap f(v)\leqq\sup_r f(y_r)\in J$. Thus we have either $f(u)\in J$ or $f(v)\in J$, i.e. $u\in(a]$ or $v\in(a]$. It follows that $(a]$ is a prime ideal. And if $\{x_\alpha | \alpha\in\Delta\}$ and there exists $\sup_{\alpha\in\Delta} x_\alpha$, then clearly $\sup_{\alpha\in\Delta} x_\alpha \in(a]$. Moreover we can see easily: $x\notin(a]$ and $f^{-1}(J)\subseteq(a]$.

(ii) J is a nonvoid dual CP-ideal. This is a dual of (i).

(iii) $J=\phi$. In this case, we may take the empty set as I. This proves our corollarly.

We recall (Funayama. [2]) that if a partially ordered set P is imbedded in a complete lattice L by a mapping θ, θ is called J-density if any element x in L can be represented as a join of elements of $\theta(P)$, that is $x=\sup_r \theta(a_r)$, where $a_r \epsilon P$. And in [2], he noted that if P is imbedded in L J-densely by θ, then $\theta(a)=\inf_r \theta(a_r)$ in L if and only if $a=\inf_r a_r$ in P.

LEMMA 2. *Let a lattice P be imbedded in a complete lattice L J-densely by* θ. *Suppose that* $\{x_\alpha | \alpha \epsilon \Delta\} \subseteq P$ *and there exists* $a=\sup_{\alpha\epsilon\Delta} x_\alpha$ *then* $\theta(a)=\sup_{\alpha\epsilon\Delta}\theta(x_\alpha)$. *Then* θ *is a continuous mapping of P into L in their CP-ideal topologies.*

PROOF. It is sufficient to show that for some CP-ideal J of L, $S=\{x \epsilon P | \theta(x) \epsilon J\}$ is also a CP-ideal of P. In fact, clearly S is a prime ideal of P. And if $\{x_\alpha | \alpha\epsilon\Delta\} \subseteq S$ and there exists $\sup_{\alpha\epsilon\Delta} x_\alpha$ in P, then we have $\sup_{\alpha\epsilon\Delta} x_\alpha \epsilon S$ because $\theta(\sup_\alpha x_\alpha) = \sup_\alpha \theta(x_\alpha) \epsilon J$. Hence S is a CP-ideal of P. And dually.

Theorem 2 of [2] and lemma 2 give us the following

THEOREM 4. *Let* ϕ *be an imbedding operator on a lattice, and* ϕ^* : $\phi^*(a)=(a]$ *be the mapping of P into* P_ϕ *such that* $\phi^*(x)=\sup_\alpha \phi^*(x_\alpha)$ *if* $x=\sup_{\alpha\epsilon\Delta} x_\alpha$ *cxists. And if f be a complete isotone of* P_ϕ *into a complete lattice L. Then P is continuously imbedded into L in their CP-ideal topologies.*

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REFERENCES

[1] G. Birkhoff, *Lattice theory,* Amer. Math. Soc. Coll. Publ. (1948)

[2] N. Funayama, *Imbedding partly ordered sets into infinitely distributive complete lattices,* Tôhoku Math. Jour. Vol. 8 (1956) pp. 54—62

[3] T. Naito, *Lattices with P-ideal topologies,* Tôhoku Math. Jour. Vol. 12 (1960) pp. 235—262