UNIFORMIZABILITY OF A TOPOLOGICAL SPACE

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1. Introduction

It is well known that a topology $\mathcal{T}$ for a set X is the uniform topology for some uniformity for X if and only if the topological space $(X, \mathcal{T})$ is completely regular.(1) F. A. Behrend has also given some necessary and sufficient conditions for the uniformizability of a topological space by means of the concept of a string.(2) In the present paper we shall also discuss on the necessary and sufficient conditions for the uniformizability based on the convergence class(3), the uniform covering system and the idea of *locally cofinal* subdirected set.

Before starting the theorem, it should be mentioned that the necessary terminology and uniform structures may be found in J. L. Kelley[1]. In particular, a uniformity for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ such that

a) Each member of $\mathcal{U}$ contains the diagonal Δ,

b) If $U \in \mathcal{U}$, then $U^{-1} \in U$,

c) If $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in U$,

d) If U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$,

e) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$,

The pair $(X, \mathcal{U})$ is a uniform space.

2. Lemmas and notations.

In the first place we are concerned with some definitions on directed sets. A directed set is defined as follows: a binary relation $\geq$ directs a set D if D is non void and

a) If $m, n, p \in D$ with $m \geq n$, $n \geq p$ then $m \geq p$,

b) If $m \in D$, then $m \geq m$,

c) If $m, n \in D$, then there is p in D such that $p \geq m$, $p \geq n$.

A directed set is a pair $(D, \geq)$ such that $\geq$ directs D.

DEFINITION 1. A subset E of a directed set $(D, \geq)$ is called a *subdirected* set of D if and only if the binary relation $\geq$ directs E.

Suppose that for each a in a set A, we are given a directed set $(D_a, >_a)$.

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(1) J. L. Kelley [1]
(2) F. A. Behrend [3]
(3) C. Y. Kim [2]

DEFINITION 2. A subset D_1 of a product directed set $(\times\{D_a : a \in A\}, \geq)$ is called *locally cofinal* if and only if each coordinate set $\{d_a : d \in D_1,\ d_a$ is the a-th coordinate of $d\}$ of D_1 is cofinal in D_a.

DEFINITION 3. Two directed sets D_1 and D_2 are *similar* if and only if there exists a one-to-one correspondence between them which preserves the order $\geq$.

We now define some notations: If the sequence $c=\{x_n\}$, $n=1,2,3,\cdots$ converges to a point x in a topological space X, then we denote it by the ordered pair $(c,x) =\{x_1,x_2,\cdots;x\}$ or briefly by c_x and $c_x(n)$ means the set of all elements which follows x_n in c, that is, $c_x(n)=\{x_i : i>n\}$. Now if X is a topological space with the topology $\mathcal{Y}$, then it can be easily scen that the following Lemmas hold.

LEMMA 1. *If $\mathcal{L}=\{c_x : x \in X\}$ is a family of all the sequences each of them converges to some point in X in the sense of $\mathcal{Y}$, then we have*

a) *For each point x in X, the neighborhood system $\mathfrak{N}(x)$ of the point x is directed by $\subset$.*

(In this case we use a symbol $>_x$ as a binary relation instead of $\subset$)

b) *If $c_x=\{x_1,x_2,\cdots;x\}\in\mathcal{L}$, then the natural number $n(c_x,N)$ is uniquely determined for each neighborhood N of x and if $N>_x N'$, then for each $c_x \in \mathcal{L}$ converging to x, $n(c_x,N)\geqq n(c_x,N')$.*

PROOF. a) is clear.

b) Let $c_x=\{x_1,x_2\cdots;\ x\}\in\mathcal{L}$, then for each neighborhood $N\in\mathfrak{N}(x)$, there are natural numbers $m(c_x,N)$ such that $c_x(m(c_x,N))\subset N$. Let $n(c_x,N)$ be the minimum of such $m(c_x,N)$'s for c_x and N, then $n(c_x,N)$ is a natural mumber and uniquely determined. And if $N>_x N'$, then $N\subset N'$ and therefore $n(c_x,N)\geqq n(c_x,N')$.

Let $\mathfrak{N}(x)=\{N(x)\}$ be the neighborhood system of x in X. Then $\{\mathfrak{N}(x),>_x\}$ is a directed set.

We now consider the product directed set $\{D,\geq\}=\times\{\mathfrak{N}(x) : x\in X\}$. Then by Lemma 1, b) the following Lemma holds.

LEMMA 2. *If $c_x=\{x_1,x_2\cdots;\ x\}\in\mathcal{L}$, then the natural number $n(c_x,d)$ is uniq-*

uely determined for each d in D, and if $d \geq d'$ in D, then for each $c_x \epsilon \mathscr{L}$, $n(c_x, d) \geqq n(c_x, d')$.

PROOF. Let $n(c_x, d) = n(c_x, d_x)$, where d_x is the x-th coordinate of d. Then by lemma 1. b), $n(c_x, d)$ is uniquely determined and if $d \geq d'$ then $n(c_x, d) \geqq n(c_x, d')$.

Now let $(X, \mathfrak{U})$ be the uniform space whose uniform topology is $\mathscr{T}$. Then for each U in $\mathfrak{U}$ and each $x \epsilon X$, $\{U[x]\}$ is the neighborhood system of x in X. Then for each U in $\mathfrak{U}$ there correspond only one member d in D, such that $U[x] = d_x$ for each x in X. Hence let D_1 be the set of $d \epsilon D$ (such that for each x in $X, d_x = U[x]$) corresponding to each U in $\mathfrak{U}$. Then we have

LEMMA 3. *$\mathfrak{U}$ is directed by $\subset$. And D_1 is the sub-directed set of D which is similar to $\mathfrak{U}$. And D_1 is locally cofinal in D.*

PROOF. If $d_1 \geq d_2$ in D_1, then for each x in X $d_{1x} >_x d_{2x}$, therefore for their corresponding elements U_1, U_2, in $\mathfrak{U}$, $U_1[x] \subset U_2[x]$ for each x in X. That is, $U_1 \subset U_2$. Hence D_1 is similar to $\mathfrak{U}$. And also since for each x in X, $\{U[x] : U \epsilon \mathfrak{U}\}$ is the neighborhood system of x, D_1 is locally cofinal in D.

Let $(X, \mathfrak{U})$ be the unifom space with the uniform topology $\mathscr{T}$, and let $\mathfrak{N}(x)$ be the neighborhood system of x in X, and let $\mathscr{L}$ be the family of all sequences converging to some points in X.

Now we denote the set $c_x(n(c_x, d)) = \{x_i : i > n(c_x, d)\}$ by $c_x(d)$ and let $A(x, d) = \cup \{c_x(d) : c_x \epsilon \mathscr{L}$, and x and d are fixed $\}$ and $B(x, d) = \{(x, y) : y \epsilon A(x, d)\}$. Then

LEMMA 4. *$B(d) = \cup \{B(x, d) : x \epsilon X\}$ is identical with U in $\mathfrak{U}$ which corresponds to d in D_1.*

PROOF. It is clear that $B(d) \subset U$. Let $(x, y) \epsilon U$. Since $c_x = \{y, y, x, x, \cdots; x\}$ converges to x relative to the uniform topology, c_x belongs to $\mathscr{L}$. And since all elements of c_x belong to $U[x] = d_x$, $n(c_x, d) = 1$, therefore $y \epsilon c_x(d)$ and $(x, y) \epsilon B(d)$ or $U \subset B(d)$. Hence $U = B(d)$.

3. Theorems.

Let $\mathscr{L}$ be the set of all sequences c_x converging to some point x in a uniform space $(X, \mathfrak{U})$. Then we have

THEOREM 1. a) *There is a locally cofinal subdirected set* D_1 *of the product directed set* $(\times\{\mathfrak{N}(x): x \in X\}, \geq)$.

b) *If* $c_x \in \mathscr{L}$, *then* $n(c_x, d)$ *is uniquely determined for each* d *in* D_1 *and if* $d \geq d'$ *in* D_1 *then* $n(c_x, d) \geqq n(c_x, d')$.

c) (i) *For each* $d \in D_1$ *and each* x *in* X, *there is a member* d' *in* D_1 *such that if* $x \in c_y(d')$ *where* $c_y \in \mathscr{L}$, *then there is* $c_x \in \mathscr{L}$ *with* $c_x(d) \ni y$.

(ii) *For each* $d \in D_1$ *and each* x *in* X, *there is a member* d' *in* D_1 *such that if* $x \in c_y(d')$ *and* $y \in c_z(d')$, *where* c_y, $c_z \in \mathscr{L}$, *then there is a sequence* c_z' *in* $\mathscr{L}$ *with* $c_z'(d) \ni x$.

PROOF. a) By lemma 3 it is clear.

b) By lemma 2 it is clear.

c) (i) Let $d \in D_1$ and $x \in X$, then there is a member U in $\mathfrak{U}$ corresponding to d. Since $\mathfrak{U}$ is the uniformity for X there is a member U^{-1} in $\mathfrak{U}$ which corresponds to d' in D_1. If x belongs to $c_y(d')$, where $c_y \in \mathscr{L}$, then $(y, x) \in B(y, d') \subset B(d') = U^{-1}$, hence $(x, y) \in U = B(d)$. And it is clear that $c_x = \{y, y, x, x, \cdots; x\}$ converges to x and $y \in c_x(d)$.

(ii) Let $U \in \mathfrak{U}$ and $x \in X$, then there is a member $V \in \mathfrak{U}$ such that $V \circ V \subset U$ for some V in $\mathfrak{U}$. Let d, $d' \in D_1$ correspond to U, V, respectively. Now if $c_z(d') \ni y$ and $c_y(d') \ni x$, where c_y, $c_z \in \mathscr{L}$, then (z, y), $(y, x) \in V$, therefore $(z, x) \in V \circ V \subset U$, or $(z, x) \in B(d)$. Hence $x \in c_z'(d)$ for some c_z' in $\mathscr{L}$.

Let X be a topological space with the topology $\mathscr{T}$ and let $\mathfrak{N}(x)$ be the neighborhood system of x in X. Then $\{N(x): N(x) \in \mathfrak{N}(x)\}$ is directed by $\subset$. We denote $\subset$ by $>_x$.

In order to clear our statements we describe some notations again. Let D be the product directed set $\times\{\mathfrak{N}(x): x \in X\}$ and let $\mathscr{L}$ be he family of all sequences each of them converges to some point in X. If the sequence $c = \{x_1, x_2, \cdots\cdots\cdots\}$ converges to $x \in X$, then we denote it by c_x. For each neighbrohood $N(x)$ of x

and each sequence $c_x=\{x_1, x_2, \cdots\cdots; x\}\epsilon\mathscr{L}$, there is a minimal number $n(c_x, N(x))$ such that if $i>n$ then $x_i \epsilon N(x)$. Let $c_x(N(x))=\{x_i : i>n(c_x, N(x))\}$ and for each $d\epsilon D$ let $c_x(d)=c_x(d_x)$, where d_x means the x-th coordinate of d.

THEOREM 2. *A topological space* $(x, \mathscr{V})$ *is uniformizable if and only if there is a locally cofinal subdirected set* D_1 *of* D *such that*

(i) *For each* d *in* D_1 *and each* x *in* X, *there is a member* d' *in* D_1 *such that if* $x \epsilon c_y(d')$ *for some* c_y *in* $\mathscr{L}$, *then* $y \epsilon c_x(d)$ *for some* $c_x \epsilon \mathscr{L}$.

(ii) *For each* d *in* D_1 *and each* x *in* X, *there is a member* d' *in* D_1 *such that if* $x \epsilon c_y(d')$ *and* $y \epsilon c_z(d')$ *for some* c_x, c_y *in* $\mathscr{L}$, *then* $x \epsilon c_z'(d)$ *for some* c_z' *in* $\mathscr{L}$.

PROOF. The necessity follows from theorem 1. We now prove the sufficiency. For $x \epsilon X$ and $d\epsilon D_1$, let $A(x,d)=\cup\{c_x(d) : c_x\epsilon\mathscr{L}\}$ and let $B(x,d)=\{(x,y): y \epsilon A(x,d)\}$ and $B(d)=\cup\{B(x,d); x \epsilon X\}$. And let $\mathfrak{U}$ be the family of all set U each of them contains $B(d)$ for some d in D_1.

We now first prove that $\mathfrak{U}$ is the uniformity for X.

(a) If $U\epsilon\mathfrak{U}$, then U contains some $B(d)=\cup\{B(x,d)\ x \epsilon X\}$, and because $c_x=\{x, x, x, \cdots; x\}$ belongs to $\mathscr{L}$, $B(x,d)$ contains (x,x) and therefore for each x in X, $(x,x)\epsilon U$, hence $U\supset\Delta$.

(b) Let $U\epsilon\mathfrak{U}$, then $U\supset B(d)$ for some d in D_1. By the condition (i) there is d' in D_1 such that if $x \epsilon c_y(d')$, then $y \epsilon c_x(d)$ for some c_x in $\mathscr{L}$. Now we show that U^{-1} contains $B(d')$. Let $(y,x)\epsilon B(d')$, then by definition of $B(d')$ there is some c_y in $\mathscr{L}$ with $x \epsilon c_y(d')$. Hence by above condition (i) $c_x(d)\ni y$ for some c_x in $\mathscr{L}$. That is, $(x,y) \epsilon B(x,d) \subset B(d) \subset U$, or $(y,x) \epsilon U^{-1}$. Therefore U^{-1} contains $B(d')$ and $U^{-1}\epsilon\mathfrak{U}$.

(c) Let $U\epsilon\mathfrak{U}$, then $U\supset B(d)$ for some d in D_1. Let d' be an element of D_1 which satisfies the condition (ii) for d. And let $B(d')=V$ and let $(x,z)\epsilon V\circ V$, then for some y in X $(x,y) \epsilon V$ and $(y,z) \epsilon V$ or (x,y), $(y,z) \epsilon B(d')$. Hence $y \epsilon c_x(d')$ for some $c_x \epsilon \mathscr{L}$ and $z \epsilon c_y(d')$ for some $c_y \epsilon \mathscr{L}$. By (ii) there is some c_x' in $\mathscr{L}$ such that $z \epsilon c_x'(d)$. That is, $(x,z)\epsilon B(d)\subset U$ or $V\circ V\subset U$.

(d) Let $U, V \epsilon \mathfrak{U}$, then for some d and d' in D_1, $B(d)\subset U$ and $B(d')\subset V$. Since D_1 is a directed set there is d'' in D_1 such that $d''\geq d$, $d''\geq d'$. Hence $B(d'')\subset B(d)$ and $B(d'')\subset B(d')$ or $U\cap V\supset B(d)\cap B(d')\supset B(d'')$. That is, $U\cap V \epsilon \mathfrak{U}$.

Now we prove that the uniform topology for $\mathcal{U}$ is identical with the topology $\mathcal{T}$. For the uniform topology $\mathcal{T}_1$ of the uniformity $\mathcal{U}$, $\{U[x] ; U \epsilon \mathcal{U}\}$ is the neighborhood system of x in X. From the construction of the uniformity $\mathcal{U}$, it is clear that the family $\{U[x] ; U \epsilon \mathcal{U}\}$ is a subfamily of the neighborhood system $\mathcal{N}(x)$ in the sense of the topology $\mathcal{T}$. Therefore it is sufficient to prove that $\{U[x] : U \epsilon \mathcal{U}\}$ is the base for $\mathcal{N}(x)$.

Let $N(x)$ be the neighborhood of x relative to the topology $\mathcal{T}$. Then there is a member d in D whose x-th coordinate d_x is $N(x)$. Since D_1 is locally cofinal, there is a member d' in D_1 such that $d_x' >_x d_x$. Then $A(x, d') = d_x' \subset N(x)$. On the other hand $\{A(x,d) ; d \epsilon D_1\}$ is the base for the neighborhood system of x relative to the uniform topology $\mathcal{T}_1$. Therefore $\{U[x] ; U \epsilon \mathcal{U}\}$ is the base for $\mathcal{N}(x)$.

In the following theorem we also give some other necessary and sufficient condition for the uniformizability of a topological space based on the uniform covering system and the idea of locally cofinal subdirected set. The uniform covering system Φ is a collection of covers of a set X such that:

(a) if $\mathcal{A}$ and $\mathcal{B}$ are members of Φ, then there is a member of Φ which is a refinement of both $\mathcal{A}$ and $\mathcal{B}$;

(b) if $\mathcal{A} \epsilon \Phi$, then there is a member of Φ which is a star refinement of $\mathcal{A}$; and

(c) if $\mathcal{A}$ is a cover of X and some refinement of $\mathcal{A}$ belongs to Φ, then $\mathcal{A}$ belongs to Φ,

And the family of all sets of the form $\cup\{A \times A : A \epsilon \mathcal{A}\}$ for $\mathcal{A}$ in Φ is a base for some uniformity $\mathcal{U}$ for X.

Now we consider the product directed set $\{D, \geq\} = \times\{\mathcal{N}(x) : x \epsilon X\}$ in lemma 2. Then each member d of D can be considered a covering of a topological space X and D is a covering system of X. It can easily seen that the above conditions (a) and (c) in the uniform covering system are satisfied for this covering system D. Now we prove the following theorem:

THEOREM 3. *A topological space $(X, \mathcal{T})$ is uniformizable if and only if there is a locally cofinal subdirected set D_1 in D such that if $d \epsilon D_1$, then there exists a member of D_1, which is a star refinement of d.*

PROOF. Sufficiency. If D_1 is the locally cofinal subdirected set of D which satisfies the condition in the theorem, then D_1 is a uniform covering system for X. Hence the family of all sets of the form $\cup\{N \times N : N \epsilon d\}$ for d in D_1 is a base for some uniformity $\mathcal{U}$ for X. And since D_1 is locally cofinal in D, the uniform topology is identical with topology $\mathcal{T}$ of X. This is precisely the situation which occurs in the proof the theorem 2.

Necessity. Let $(X, \mathcal{U})$ be a uniform space. Then $\mathcal{U}$ is a directed set and the family $\{U[x]: U \in \mathcal{U}\}$ is the neighborhood system of x and by lemma 3 there is a locally cofinal subdirected set D_1 in $D = \times \{\mathcal{N}(x): x \in X\}$ which is similar to $\mathcal{U}$, where $\mathcal{N}(x) = \{U[x]: U \in \mathcal{U}\}$. We now prove that the covering system D_1 of X satisfies the condition (b) in the uniform covering system. Let $d \in D_1$, then there exists a member U in $\mathcal{U}$ which corresponds to d. Since $\mathcal{U}$ is the uniformity for X, there exists a symmetric member V in $\mathcal{U}$ such that $V \circ V \circ V \subset U$. Let d' in D_1 be the member corresponding to V, then for each x in X, $V[x]$ is a x-th coordinate of d , and d' is a covering of X whose members are $\{V[x]: x \in X\}$. Let $V[y]$ be any member of covering d' with $V[y] \cap V[x] \neq 0$ and let $z \in V[y]$, then $(x, z) \in V \circ V \circ V \subset U$ because of the symmetricity of V. Hence $V[y] \subset U[x]$, that is, d' is the star refinement of d.

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