ON DEFINITIONS OF A UNIFORM SPACE BY THE CONVERGENCE CLASS

By Chi Young Kim

The various methods of describing the uniform space have been investigated until now. In 1937 *A. Weil* has defined the uniform space by means of the uniform neighborhood system in his paper [2][1]. After that *J. W. Tukey* also investigated the same uniform structure using a uniform covering systems in his paper [3] in 1940. On the other hand *J. L. Kelley* descirbed the uniformity for a set X in his book [1] as follows: A uniformity for a set X is a non-void family $\mathscr{U}$ of subsets of $X \times X$ such that, (a) each member of $\mathscr{U}$ contains the diagonal $\triangle$, (b) If $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$, (c) If $U \in \mathscr{U}$, then $V \circ V \subset U$ for some V in $\mathscr{U}$, (d), If U and V are members of $\mathscr{U}$, then $U \cap V \in \mathscr{U}$, and (e) If $U \in \mathscr{U}$ and $U \subset V \subset X \times X$, then $V \in \mathscr{U}$. In this paper I have tried to define the same uniform structure by a convergence class on the bases of *Weil*'s uniform neighborhood system and on the unifomity $\mathscr{U}$ of *Kelley*.

Before stating the theorem, it should be first mentioned that the necessary terminology and uniform structures may be found in *Kelley* [1]. And we now define some notations: If the sequence $C = \{x_n\}$, $n = 1, 2, \cdots$ is converges to x relative to the uniform topology, then we denote it by C_x and $C(n)$ means the set of all elements which follow x_n in C, i.e. $C(n) = \{x_i : i > n\}$. Now if X is a uniform space, the uniformity of which is $\mathscr{U} = \{U\}$, then it can be easily seen that the following lemmas hold.

LEMMA 1. *If $\mathscr{L}$ is a family of all the sequences each of them converges to some point in X in the sence of uniform topology, then we have*

1) Numbers in brackets represent references listed at the end of the paper.

a) If $C=\{x_1, x_2, \cdots, x_m, \cdots\}$, where $x_{n+1}=x_{n+2}=\cdots=x$ then $C\in \mathcal{L}$ and C converges to x.

b) The uniformity $\mathcal{U}$ for X is directed by $\subset$.

(In this case we use a symbol $\geq$ as a binary relation instead of $\subset$.)

c) If $C=\{x_1, x_2, \cdots\}\in\mathcal{L}$, then the natural number $n(C,U)$ is uniquely determined for each U in $\mathcal{U}$, and if $U\leq V$ ($\in\mathcal{U}$), then for each C in $\mathcal{L}$ $n(C,U)\leqq n(C,V)$.

PROOF. *a*) Since X is a uniform space, $C=\{x_1, \cdots, x_n, x, x, \cdots\}$ converges to x relative to the uniform topology. Hence $C\in\mathcal{L}$.

b) is clear.

c) Let $C_x=\{x_1, x_2, \cdots\}$ converges to x in the sence of uniform topology, then for each $U\in\mathcal{U}$ there are natural numbers $m(C_x,U)$ such that $C_x(m(C_x,U))\subset U[x]$ because $U[x]$ is a neighborhood of x relative to the uniform topology. Let $n(C_x,U)$ be the minimum of such $m(C_x,U)$'s for C_x and U, then $n(C_x,U)$ is a natural number and is uniquely determined. And if $U\leq V$, then $U[x]\supset V[x]$ and therefore $n(C_x,U)\leqq n(C_x,V)$.

Now we denote the set $C_x(n(C_x,U))=\{x_i : i>n(C_x,U)\}$ by $C_x(U)$ and let $A(x,U)=\bigcup\{C_x(U)\colon C_x\in\mathcal{L}$, and x and U are fixed$\}$ and $B(x,U)=\{(x, y)\colon y\in A(x, U)\}$, then we have following lemma 2 under the same conditions as in the case of lemma 1.

LEMMA 2. *d*) $B(U)=\bigcup\{B(x, U) : x\in X\}$ *is identical with* $U\in\mathcal{U}$.

e) *For each* U *in* $\mathcal{U}$ *and each* x *in* X *there is a member* V *of* $\mathcal{U}$ *such that if* $x\in C_y(V)$ *where* $C_y\in\mathcal{L}$, *then there is a sequence* C_x *in* $\mathcal{L}$ *with* $C_x(U)\ni y$.

f) *For each* $U\in\mathcal{U}$ *and each* x *in* X, *there is a member* V *in* $\mathcal{U}$ *such that if* $x\in C_y(V)$ *and* $y\in C_z(V)$, *where* C_z, $C_y\in\mathcal{L}$, *then there is a sequence* C_z' *in* $\mathcal{L}$ *with* $C_z'(U)\ni x$.

PROOF. *d*) It is clear that $B(U)\subset U$. Let $(x, y)\in U$. Since $C_x=\{y, y, x, x, x, \cdots\}$ converges to x relative to the uniform topology, $C_x\in\mathcal{L}$. And since all the elements of C_x belong to U x , $n(C_x, U)=1$, therefore $y\in C_x(U)$ and $(x, y)\in B(U)$, or $U\subset B(U)$. Hence $U=B(U)$.

e) Let $U \in \mathfrak{U}$ and $x \in X$, then $U^{-1}=V \in \mathfrak{U}$. If x belongs to $C_y(V)$, where C_y is in $\mathscr{L}$, then $(y, x) \subset B(y, V) \subset B(V)=V$, hence $(x, y) \in V^{-1}=U=B(U)$. And it is clear that $C_x=\{y, y, x, x, x, \cdots\}$ converges to x and $y \in C_x(U)$.

f) Let $U \in \mathfrak{U}$ and $x \in X$, then there is a member $V \in \mathfrak{U}$ such that $V \circ V \subset \mathfrak{U}$ for some V in $\mathfrak{U}$. If $C_z(V) \ni y$, and $C_y(V) \ni x$, where C_z and $C_y \in \mathscr{L}$, then (z, y), $(y, x) \in V$, therefore $(z, x) \in V \circ V \subset U$, or $(z, x) \in B(U)$. Hence $x \in C_z'(U)$ for some C_z' in $\mathscr{L}$.

From the preceding discussion of convergence we know several properties which must hold for $\mathscr{L}$. So we now consider a family $\mathscr{L}=\{C\}$ of sequences in X and directed set$(D, \geq)$ and a function N on the cartesian product $\mathscr{L} \times D$ into the set of all natural numbers, and we shall say that the triplet $(\mathscr{L}, D, N)$ is a convergence class for X if and only if it satisfies the following conditions. For convenience, we say that $C=\{x_n\}$ converges $(\mathscr{L})$ to x or that $\lim_{n\to\infty} x_n \equiv x\,(\mathscr{L})$ if and only if $C \in \mathscr{L}$, and when C converges $(\mathscr{L})$ to x we denote it by C_x.

i) If $C=\{x_n\}$ is a sequence such that $x_n=x$ for each n, then $C \in \mathscr{L}$.

ii) Relation $\geq$ directs the set D and the range of a function N on $\mathscr{L} \times D=\{(C, d)\}$ is the set of natural numbes, and if $d' \geq d$, then $N(C, d') \geqq N(C, d)$.

iii) For each d in D and each x in X, there is a member d' in D such that if $x \in C_y(N(C_y, d'))$ for C_y in $\mathscr{L}$, then $y \in C_x(N(C_x, \mathrm{d}))$ for some C_x in $\mathscr{L}$.

iv) For each d in D and each x in X there is a member d' in D such that if $x \in C_y(d')$ and $y \in C_z(d')$ for some C_y, C_z in $\mathscr{L}$ then $x \in C_z'(d)$ for some C'_z in $\mathscr{L}$. (where $C_x(d)=C_x(N(C_x, d))$, etc.)

We will now proceed to show that a uniform structure is defined by the convergence class as follows. In order to clear our statements we describe some notations again.

For $x \in X$ and $d \in D$, let $A(x, d)= \bigcup\{C_x(d): C_x \in \mathscr{L}\}$ and $B(x, d) =\{(x, y): y \in A(x, d)\}$, and $B(d)=\bigcup\{B(x, d): x \in X\}$.

THEOREM 1. *Let$(\mathscr{L}, D, N)$ be a convergence class for a set*

X, and let $\mathfrak{U}$ be the family of all sets U each of them contains $B(d)$ for some d in D, then $\mathfrak{U}$ is a uniformity for X and each $C \in \mathcal{L}$ converges to its limit relative to the uniform topology.

PROOF. We now prove that $\mathfrak{U}$ is a uniformity for X. For this purpose we must prove that $\mathfrak{U}$ satisfies *Kelley*'s four conditions (a), (b), (c) and (d) mentioned above.

(a) If $U \in \mathfrak{U}$, then U contains some $B(d) = \cup \{B(x, d): x \in X\}$, and because $C = \{x, x, \cdots\}$ belongs to $\mathcal{L}$, $B(x, d)$ contains (x, x), therefore for each x in X $(x, x) \in U$, hence $U \supset \triangle$.

(b) Let $U \in \mathfrak{U}$, then $U \supset B(d)$ for some d in D. By the condition iii) there is d' in D such that if $x \in C_y(d')$, then $y \in C_x(d)$ for some C_x in $\mathcal{L}$. Now we show that U^{-1}contains $B(d')$. Let $(y, x) \in B(d')$ then by definition of $B(d')$ there is some C_y in $\mathcal{L}$ with $x \in C_y(d')$. Hence by above condition iii) $C_x(d) \ni y$ for some C_x in $\mathcal{L}$. That is, $(x, y) \in B(x, d) \subset B(d) \subset U$, or $(y, x) \in U^{-1}$. Therefore U^{-1} contains $B(d')$ and $U^{-1} \in \mathfrak{U}$.

(c) Let $U \in \mathfrak{U}$, then $U \supset B(d)$ for some d in D. Let d' be an element of D which satisfies the condition iv) for d. And let $B(d') = V$ and let $(x, z) \in V \circ V$, then for some y in X $(x, y) \in V$ and $(y, z) \in V$, or $(x, y) \in B(d')$ and $(y, z) \in B(d')$. Hence $y \in C_x(d')$ for some C_x in $\mathcal{L}$ and $z \in C_y(d')$ for some C_y in $\mathcal{L}$. By iv) there is some C'_x in $\mathcal{L}$ such that $z \in C'_x(d)$; that is, $(x, z) \in B(d) \subset U$ or $V \circ V \subset U$.

(d) Let U, $V \in \mathfrak{U}$, then for some d and d' in D, $B(d) \subset U$ and $B(d') \subset V$. Since D is a directed set there is d'' in D such that $d'' \geq d$ and $d'' \geq d'$. Hence $B(d'') \subset B(d)$ and $B(d'') \subset B(d')$ or $U \cap V \supset B(d) \cap B(d') \supset B(d'')$. That is, $U \cap V \in \mathfrak{U}$.

(e) By the definition of $\mathfrak{U}$ it is clear that if $U \in \mathfrak{U}$ and $U \subset V \subset X \times X$. then $V \in \mathfrak{U}$. Therefore $\mathfrak{U}$ is a uniformity for X. The last part of the theorem follows from the following theorem 2.

If X is a uniform space whose convergence class is $(\mathcal{L}, D, N)$, then we can also induce a uniform topology $\mathcal{T}$ in the following way. Let $\mathcal{T}$ be the family of all subsets T of X such that for each $x \in T$ there is some d in D with $A(x, d) \subset T$. Then $\mathcal{T}$ is indeed a uniform topology for X which is derived from the uniformity $\mathcal{U}$, for $\{B(d):d \in D\}$ is a base of $\mathcal{U}$ and $A(x, d)=B(d)[x]$. Now we prove the following theorem directly by means of i)~iv).

THEOREM 2. *The interior of a subset M of X in the sence of uniform topology is the set of all points x with $A(x, d)\subset M$ for some d in D. And for each x in X, the family $\{A(x,d) : d \in D\}$ is a base for the neighborhood system of x.*

PROOF. In order to prove that a set $M^i=\{x: A(x, d)\subset M$ for some d in $D\}$ is an interior of M, it is sufficient to show that the set M^i is open in X relative to the uniform topology because M^i is a maximal open subset of M if it is open. If $x \in M^i$, then there is some d in D such that $A(x, d)\subset M$ and by iv) there is d' in D such that the condition iv) holds for d and d'. If $y \in A(x, d')$, then there is a sequence C_x in $\mathcal{L}$ such that $y \in C_x(d')$. And if $z \in A(y, d')$, then there is a sequence C_y in $\mathcal{L}$ such that $z \in C_y(d')$. By iv) there is some C_x' in $\mathcal{L}$ with $z \in C'_x(d)$; that is, $z \in A(x, d)$. This means that $A(y, d') \subset A(x, d) \subset M$. Hence $y \in M^i$, or $A(x, d')\subset M^i$ and M^i is open. Now we show that $\{A(x, d): d \in L\}$ is a base of the neighborhood system of x. The interior of $A(x, d)$ is not void since it contains at least x. Hence $A(x, d)$ contains an open set to which x belongs, and it is a neighborhood of x. Since for each $x \in X$ it is clear that every neighborhood of x contains some $A(x, d)$ for some d in D, $\{A(x, d):d \in D\}$ is a base for the neighborhood system of x.[2)]

It is easily seen that if the convergence class $(\mathcal{L}, D, N)$ of X satisfies the following additional condition v), then the sequence C converges to some point of X relative to the topolpgy $\mathcal{T}$ if and only if $C \in \mathcal{L}$.

v) If C is a sequence such that for each d in D there is n with $C(n) \subset A(x, d)$, then C belongs to $\mathcal{L}$ and it's limit is x.

Then we have the followings.

2) The proof of theorem 2 is much due to *Kelley* [1]. See *Kelley*[1], chap. 6, theo. 4.

THEOREM 3. *If* $(\mathscr{L}, D, N)$ *is a convergence class for* X, *then there is a uniformity* $\mathscr{U}$ *for* X *such that a sequence* C *converges relative to the uniform topology if and only if* $C \in \mathscr{L}$.

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Department of Mathemetics
Korea University

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