ON THE CONFORMAL CORRESPONDENCE BETWEEN THE FIRST AND SECOND PARAMETER-GROUP SPACES

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1. Introduction.

On a continuous transformation group with n independent variables and r essential parameters a, let the equation of the parameter-group be

$$a_3^{\alpha}=\varphi^{\alpha}(a_1, a_2) \qquad (\alpha=1, 2, \cdots\cdots, r). \tag{1. 1}$$

For this equation, the fundamental differential equations of the first and second parameter-groups of the given continuous transformation group are represented by

$$\begin{cases} \dfrac{\partial a_3^{\alpha}}{\partial a_2^{\beta}}=A_b^{\alpha}(a_3)\, A_\beta^{b}(a_2), \\ \dfrac{\partial a_3^{\alpha}}{\partial a_1^{\beta}}=\bar{A}_b^{\alpha}(a_3)\, \bar{A}_\beta^{b}(a_1), \qquad (b,\ \alpha,\ \beta=1, 2, \cdots\cdots, r) \end{cases} \tag{1. 2}$$

Furthermore, We define the group-spaces with connections

$$\begin{aligned} L_{\beta\gamma}^{\alpha} &= A_b^{\alpha}\frac{\partial A_\beta^{b}}{\partial a^{\gamma}} = -A_\beta^{b}\frac{\partial A_b^{\alpha}}{\partial a^{\gamma}}, \\ \bar{L}_{\beta\gamma}^{\alpha} &= \bar{A}_b^{\alpha}\frac{\partial \bar{A}_\beta^{b}}{\partial a^{\gamma}} = -\bar{A}_\beta^{b}\frac{\partial \bar{A}_b^{\alpha}}{\partial a^{\gamma}} \end{aligned} \tag{1. 3}$$

and

$$\Gamma_{\beta\gamma}^{\alpha}=\frac{1}{2}(L_{\beta\gamma}^{\alpha}+\bar{L}_{\beta\gamma}^{\alpha})=\frac{1}{2}(L_{\beta\gamma}^{\alpha}+L_{\gamma\beta}^{\alpha}), \tag{1. 4}$$

and we have been denoted by $S^{(+)}$, $S^{(-)}$ and $S^{(0)}$ respectively.

Now we introduce the metric tensors $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ as the following in $S^{(+)}$ and $S^{(-)}$ respectively;

$$g_{\alpha\beta}=A_\alpha^{a}A_\beta^{a},\ \ \bar{g}_{\alpha\beta}=\bar{A}_\alpha^{a}\bar{A}_\beta^{a},$$

and want to seek a necessary and sufficient condition in order that there exists a conformal correspondence between the group-spaces $S^{(+)}$ and $S^{(-)}$.

The fundamental quantities for these spaces were studied by Nobuo Horie[1]. Here, we use of his results in the paper[1].

The Christoffel symbol with respect to $g_{\alpha\beta}$ is represented by

(1. 5) $$\left\{{\alpha \atop \beta\gamma}\right\}=\Gamma^{\alpha}_{\beta\gamma}+\frac{1}{2}(c^{e}_{ab}+c^{b}_{ae})A^{\alpha}_{a}A^{e}_{\beta}A^{b}_{\gamma},$$

where c^{e}_{ab} are the constants of structure of $S^{(+)}$.

2. The conformal correspondence between $S^{(+)}$ and $S^{(-)}$.

Let $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ denote the metric tensors of $S^{(+)}$ and $S^{(-)}$, then the conformal correspondence between the group-spaces $S^{(+)}$ and $S^{(-)}$ is defined by

$$\bar{g}_{\alpha\beta}=e^{2\sigma}g_{\alpha\beta}.$$

If there exists the conformal correspondence between $S^{(+)}$ and $S^{(-)}$, then it satisfies the equations

(2. 1) $$\overline{\left\{{\alpha \atop \beta\gamma}\right\}}=\left\{{\alpha \atop \beta\gamma}\right\}+\delta^{\alpha}_{\beta}\sigma_{,\gamma}+\delta^{\alpha}_{\gamma}\sigma_{,\beta}+g_{\beta\gamma}g^{\alpha\lambda}\sigma_{,\lambda},$$

and conversely, if this differential equations are completely integrable, then there exists the correspondence.

From (1. 5) and $\Gamma^{\alpha}_{\beta\gamma}=\bar{\Gamma}^{\alpha}_{\beta\gamma}$, (2. 1) is reduced to

$$\frac{1}{2}(\bar{c}^{e}_{ab}+\bar{c}^{b}_{ae})\bar{A}^{\alpha}_{a}\bar{A}^{e}_{\beta}\bar{A}^{b}_{\gamma}$$
$$=\frac{1}{2}(c^{e}_{ab}+c^{b}_{ae})A^{\alpha}_{a}A^{e}_{\beta}A^{b}_{\beta}+\delta^{\alpha}_{\beta}\sigma_{,\gamma}+\delta^{\alpha}_{\gamma}\sigma_{,\beta}+g_{\beta\gamma}g^{\alpha\lambda}\sigma_{,\lambda}.$$

In order to adjust for $\sigma_{,\gamma}$, contracting with respect to α and β, since $\bar{c}^{b}_{aa}=c^{b}_{aa}=0$, we have

(2. 2) $$\sigma_{,\gamma}=\frac{1}{2(n+2)}(\bar{c}^{a}_{ab}\bar{A}^{b}_{\gamma}-c^{a}_{ab}A^{b}_{\gamma}).$$

From Maurer-Cartan equation, i. e.,

$$c^{e}_{ab}=(L^{\alpha}_{\beta\gamma}-L^{\alpha}_{\gamma\beta})A^{e}_{\alpha}A^{\beta}_{a}A^{\gamma}_{b},$$

we obtain $c^{a}_{ab}=(L^{\alpha}_{\alpha\gamma}-L^{\alpha}_{\gamma\alpha})A^{\gamma}_{b}$, and consequently, substituting it in (2. 2), (2. 2) is reduced to

$$\sigma_{,\gamma}=\frac{1}{2(n+2)}\{(\bar{L}^{\alpha}_{\alpha\gamma}-\bar{L}^{\alpha}_{\gamma\alpha})-(L^{\alpha}_{\alpha\gamma}-L^{\alpha}_{\gamma\alpha})\}.$$

Since $\bar{L}^{\alpha}_{\alpha\gamma}=L^{\alpha}_{\gamma\alpha}$, we have

$$\sigma_{,\gamma}=\frac{1}{n+2}(L^{\alpha}_{\gamma\alpha}-L^{\alpha}_{\alpha\gamma}),$$

and furthermore, from Maurer-Cartan equation, it is reduced to

(2. 3) $$\sigma_{,\gamma}=\frac{1}{n+2}c^{e}_{ae}A^{a}_{\gamma}.$$

From the completely integrable condition $\partial\sigma_{,\gamma}/\partial a^{\beta}=\partial\sigma_{,\beta}/\partial a^{\gamma}$, it must satisfy the form

$$\frac{1}{n+2}\left(\frac{\partial A^a_\beta}{\partial a^\gamma}-\frac{\partial A^a_\gamma}{\partial a^\beta}\right)c^e_{ae}=0.$$

Using of the Maurer-Cartan equation, i. e.,

$$\frac{\partial A^a_\beta}{\partial a^\gamma}-\frac{\partial A^a_\gamma}{\partial a^\beta}=c^a_{bc}A^b_\beta A^c_\gamma,$$

we have

$$c^a_{bc}c^e_{ae}A^b_\beta A^c_\gamma=0,$$

and since $|A^b_\beta| \neq 0$, then we obtain the condition

$$c^a_{bc}c^e_{ae}=0.$$

Hence, we have the following theorem:

THEOREM *A nesessary and sufficient condition that there exists a conformal correspondence between the first and second parameter-group spaces, is that*

(2. 4) $$c^a_{bc}c^e_{ae}=0.$$

Next, we consider the path with respect to affine connection, i. e.,

$$\frac{d^2a^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{da^\beta}{ds}\frac{da^\gamma}{ds}=0,$$

and in order to coincide it with the geodesic in group-space $S^{(+)}$, i. e.,

$$\frac{d^2a^\alpha}{ds^2}+\left\{{\alpha \atop \beta\gamma}\right\}\frac{da^\beta}{ds}\frac{da^\gamma}{ds}=0,$$

it must satisfy identically that

$$\left(\left\{{\alpha \atop \beta\gamma}\right\}-\Gamma^\alpha_{\beta\gamma}\right)\frac{da^\beta}{ds}\frac{da^\gamma}{ds}=0.$$

Since $\left\{{\alpha \atop \beta\gamma}\right\}-\Gamma^\alpha_{\beta\gamma}$ is symmetric with respect to β and γ, it must vanish, and consequently, from (1. 5), we have

$$\frac{1}{2}(c^e_{ab}+c^b_{ae})A^\alpha_a A^e_\beta A^b_\gamma=0.$$

Since $|A^\alpha_a| \neq 0$, we get

$$c^e_{ab}+c^b_{ae}=0,$$

i. e.,

(2. 5) $$c^e_{ab}=-c^b_{ae}.$$

Conversely, if (2. 5) are held, the path with respect to $\Gamma^\alpha_{\beta\gamma}$ and geodesic in $S^{(+)}$ are coincide with each other. Hence we have the following result:

COROLLARY *If the path with respect to* $\Gamma^{\alpha}_{\beta\gamma}$ *and the geodesic in* $S^{(+)}$ *are coincide with each other, then* $S^{(+)}$ *and* $S^{(-)}$ *are correspond conformally each other.*

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