NOTE ON A MAPPING OF CONNECTED COMPONENTS

By Mi-Soo Pae

Let f be one-to-one bicontinuous mapping between a topological space X and a topological space Y. Then X and Y are homeomorphic, so that there exists one-to-one correspondence between connected components of X and connected components of Y, by f. But there is the another condition for this matter, where f is not a homeomorphic mapping.

The correspondence f is closed if A a closed subset of X then $f(A)$ is closed in Y; f is connected if C a connected subset of X implies $f(C)$ is a connected subset of Y. [2]

For mapping f, if $f^{-1}$is closed then f is continuous mapping from X to Y. [1]

LEMMA 1. *f is closed (preimage of f), then f is connected.* [1]

We assume the following property for mapping f:

(P) For each closed subsets A, B, $f(A)\cap f(B)\neq\phi$
there exists at least one element $y\in f(A)\cap f(B)$
such that there is some $x\in f^{-1}(y)$ having sequences
$\{a_n\}\subset A$, $\{b_n\}\subset B$, $a_n\to x$, $b_n\to x$.

LEMMA 2. *f is closed mapping having property (P). If A, B are closed sets: $A\cap B=\phi$,* and *then $f(A)\cap f(B)=\phi$.*

Proof. If $f(A)\cap f(B)\neq\phi$, there exists a $x\in A\cap B$, by condition (P). Since $A\cap B=\phi$ and A, B are closed sets, then $f(A)\cap f(B)=\phi$,

LEMMA 3. *f is closed mapping with property (P). For any set M of Y, if A, B are closed sets of X such that $A\cap B\cap f^{-1}(M)=\phi$, then $f(A)$, $f(B)$ are disjoint in M.*

Proof. If it is not so: $f(A)\cap f(B)\cap M\neq\phi$. There is a x belonging to $f^{-1}(y)$, such that $a_n\to x$, $b_n\to x$, $\{a_n\}\subset A$, $\{b_n\}\subset B$ and $x\in f^{-1}(M)$, because

$$\begin{aligned} f^{-1}(y)\subset f^{-1}[f(A)\cap f(B)\cap M] &= f^{-1}[f(A)\cap M]\cap f^{-1}[f(B\cap M] \\ &= f^{-1}[f(A)]\cap f^{-1}[f(B)]\cap f^{-1}(M). \end{aligned}$$

Hence $x \in \bar{A} \cap \bar{B} \cap f(M) = A \cap B \cap f^{-1}(M)$.

Since A, B is disjoint in $f(M)$, then $f(A)$, $f(B)$ disjoint in M.

LEMMA 4. *f is biclosed mapping with properdy (P) and C is a connected component of X, then $f(C)$ is connected component of Y.*

Proof. f C) is connected by lemma 1. For any connected subset C' of Y containing $f(C)$, $f^{-1}(C')$ contains C.

Now A, B are closed sets of X such that $f^{-1}(C') \subset A \cup B$, and A, B are disjoint in $f^{-1}(C')$, then by using lemma 3, $f(A)$, $f(B)$ are disjoint in C'. $f(A)$, $f(B)$ are closed, for f is closed mapping, and $f(A) \cup f(B) \supset C'$. Since C' is connected set in Y, $f(A) \cap C' = \phi$ or $f(B) \cap C' = \phi$. Then $A \cap f^{-1}(C') = \phi$, or $B \cap f^{-1}(C') = \phi$, for $A \cap f^{-1}(C') \subset f^{-1}[f(A)] \cap f^{-1}(C') = f^{-1}[f(A) \cap C'] = \phi$.

Hence $f^{-1}(C')$ is connected. Then $f^{-1}(C')$ is equal to C, because C is a connected component of X and C is contained in the connected set $f^{-1}(C')$. So that C' is equal to $f(C)$, i. e. $f(C)$ is a connected component of Y.

REMARK. *f is closed mapping with property (P), then if M disconnected in X implies $f(M)$ is a disconnected set of Y.*

THEOREM. *For two topological spaces X, Y, if there exists biclosed mapping with property (P), then there is one-to-one correspondence between connected components of X and connected components of Y.*

Proof. Let us put $X = \bigcup_{\alpha \in I} C_\alpha$, $Y = \bigcup_{\beta \in J} C'_\beta$, where C_α is a connected component of X for all α belonging to I, and C'_β is a connected component of Y for all β belonging to J. For each α belonging to I there is a β belonging to J, such that $f(C_\alpha) = C'_\beta$ by lemma 4. Refering the above remark, $f^{-1}(C'_\beta)$ is connected. And for the connected set C containing $f^{-1}(C'_\beta)$, $f(C)$ is contained in C'_β, since C'_β is a connected component. Then $f^{-1}(C'_\beta)$ is a connected component of X. So there is an α belonging to I such that $f^{-1}(C'_\beta) = C_\alpha$ for each β of J. Finally, if C_1, C_2 are two connected components of X, then C_1, C_2 are closed and disjoint in X. By lemma 2, $f(C_1)$, $f(C_2)$ are disjoint component of Y. Hence there is one-to-one correspondence between the elements of I and the elements of J.

In this note f is continuous closed mapping and f is not one-to-one mapping.

May. 30, 1958
Mathematical Department
Liberal Arts and Science College
Kyungpook University

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[2] Compact mapping, Edwin Halfer, Proceeding of the A. M. S. Vol. 8, No. 4, 1957. pp. 828—830.