THE INTERVAL TOPOLOGY OF A LATTICE ORDERED GROUP

By Tae Ho Choe

Introduction. A group L is called a lattice ordered group, (=l-group), when in L is defined a order $a \leqq b$, preserved under the group multilication:

$$a \leqq b \text{ implies } ac \leqq bc \text{ and } ca \leqq cb \text{ for all } c \text{ in } L.$$

Let L be a partly ordered. In L, we write

$$M(x) = \{a : x \leqq a\}, \qquad L(x) = \{a : a \leqq x\} \text{ for each } x \in L.$$

The interval topology [1] is that topology generated by taking all of the sets $\{M(x),\ L(x) : x \in L\}$, as a subbasis for the closed sets.

E. S. Northam [2] has proved, using an example, that a l-group need not to be a topological group in it's interval topology, which solves problem 104 of Birkhoff [1].

In this paper, using the necessary and sufficient condition that a lattice be Hautdorff in it's interval topology, which is proved by R. M. Bare [3], we shall prove any l-group which satisfies the chain condition and not a cyclic group is not a topological group in it's interval topology. This theorem supplies an other answer to problem 104 of Birkhoff [1].

And using the necessary and sufficient condition for an element to be isolated in the interval topology of a lattice which is proved by E. S. Northam, we prove that l-group is discrete in the interval topology if and only if the l-group is isomorphic with the ordered group of all integers under addition.

In § 2, we shall prove a l-group is complete if and only if every closed interval $[e, x]$ is compact in it's interval topology. Finally, in § 3, we add that natural mapping from l-group L to the factor l-group L/H is a closed mapping in their interval topologies, where H is an l-ideal of L.

§ 1. Two Theorems.

Let L be a partly ordered set. An element of L is minimal element if it has no proper predecessor.

In a l-group L, an element a is called positive if $a \geqq e$, where the element e is an identity of group L. The set of all positive element of L is called positive cone which will always be denoted P.

Let us say that l-group is satisfying chain condition, if every non-void set of positive element includes a minimal member.

The following theorems are due to G. Birkhoff [1] and R. M. Bare [3], respectively

[THEOREM. G] *Let L be l-group which satisfies the chain condition. Then L is commutative, and each non-zero element of L can be expressed uniquely as a product of integral poweres of a finite number of distinct primes. Such a product is positive if and only if all powers are positive.*

[THEOREM. R] *A necessary and sufficient that the interval topology of a lattice L be Hausdorff is that, for every pair of elements a, b in L with $a<b$, there exist finite non-void subsets A and B (depending on a, b) in L such that both of the following conditions are satisfied*

(i) $A=\{x : a<x\leqq b$ *or* $a\#x$ *or* $b\#x\}$, $B=\{y : a\leqq y<b$ *or* $a\#y$ *or* $b\#y\}$.

(ii) $(M(x))_{x\in A}$, $(L(y))_{y\in B}$ *is a covering of* L.

Let L be a l-group which satisfies the chain condition. We suppose that L be Hausdorff in it's interval topology. Then, by Theorem R, there exist finite non-void subset A and B such that (i) and (ii) hold. And, by Theorem G, each element of A and B can be expressed uniquely as a product of integral powers of a finite number of distinct primes. Let n be the greatest integer of their all integral and zero powers, and m be the least integer of them. And we have $n+1>0$, and $m-1<0$.

Now consider the element $p^{n+1}q^{m-1}$ for some prime elements p, q if exist. Since L is a group, this element is in L. However, the element $p^{n+1}q^{m-1}$ is incomparable with any element of A and B. For, if the

element is comparble for some element of A or B, say $p^{n+1}q^{m-1} \gtrsim x$, and if the primes p and q are contained to primes of x. Then the element $p^{n+1}q^{m-1}\ x^{-1}$ is a positive. However, the power of p is positive, and that of q is negative, which is contrary. Similarly, we see easily that for any case happen the contradiction. Therefore, the element $p^{n+1}q^{m-1}$ is not in the covering (ii) of L.

Hence we have the following,

[LEMMA 1] *Let L be a l-group which satisfies the chain condition and not a cyclic group. Then L is not to be Hausdorff in it's interval topology.*

Moreover, we have

[THEOREM 1] *Under the hypotheses of Lemma 1, l-group L is not to be a topological group in it's interval topology.*

[PROOF] If the l-group L is to be a topological group. Since the interval topology is T_1-space and a topological group is regular, the interval topology must be Hausdorff, which is contrary to the Lemma 1.

E. S. Northam [2] proved the following

[THEOREM. N] *A necessary and sufficient condition for an element x to be isolated in the interval topology of a lattice L is that*

(a) x covers a finite number of elements and every element under x is under an element covered by x.

(b) x is covered by a finite number of elements and every element over x is over an element which covers x.

(c) x belongs to a finite separating set of L in which no other member is comparable with x.

Now we apply the Theorem. N to the following theorem.

[THEOREM 2] *A l-group L be discrete in it's interval topology if and only if L is isomorphic with the ordered group of all integers under addition.*

[PROOF] Let L be discrete. By Theorem N, there are a finite number of primes and a finite separating set of L, since an identity is

an isolated point of L.

From ([5], Theorm 3), it follows that L is a free abelian group with the primes as generators and every element of the finite separating set is expressed uniquely as a product of integral powers of the finite number of distinct generators. Let n be the greatest integer of their all integral and zero powers, and m be the least integer of them. Similarly, as was done Theorem 1, we can see easily that the element $p^{n+1}q^{m-1}$ for some genertors p, q if exist, is incomparable with any element of the finite separating set *of* L, which is unreasonable. Hence L is a cyclic group. And hence, L is isomorphic with the ordered group of all integers under addition.

The converse is obvious.

§ 2. Completeness of l-group.

A l-group is called complete if and only if every non-void bounded set has a g.l.b. and a l.u.b.

[LEMMA 2] *A l-group is to be complete if and only if any non-void bounded subset of the positive cone P has a g.l.b.*

[PROOF] Let M be a non-void bounded subset of L, and the element z be a lower bound of M : $z \leqq a$ for any element a of M. i.e., $e \leqq z^{-1}a$ for any element a of M. Therefore, the set $z^{-1}M = \{ z^{-1}a : a \in M \}$ is a non-void bounded subset of the positive cone P. Since in P, any non-void bounded subset has a g.l.b., the set $z^{-1}M$ has a g.l.b. x. Then the element zx is a g.l.b. of M. For, from $z^{-1}a \geqq x$, we have $a \geqq zx$ for any element a of M. Let y be a lower bound of M, then $z^{-1}y \leqq z^{-1}a$ for any element $z^{-1}a$ of $z^{-1}M$. Therefore, we have $z^{-1}y \leqq x$, i.e., $y \leqq zx$. Hence, any non-void bounded subset of L has a g.l.b.

Now consider the set $M^{-1} = \{ a^{-1} : a \in M \}$. Since M is a non-void bounded, M^{-1} is non-void bounded. And let the element v be a g.l.b. of M^{-1}. Then the element v^{-1} is a l.u.b. of M.

For, from $v \leqq a^{-1}$, we have $v^{-1} \geqq a$ for any element a of M. Let u be an upper bound of M, then $u^{-1} \leqq a^{-1}$ for any element a^{-1} of M^{-1} Therefore, we have $u^{-1} \leqq v$, i.e., $u \geqq v^{-1}$.

Now, we prove the following Lemma in a manner parallel to ([4], Theorem 1).

[LEMMA 3] *Any non-void bounded subset of the poitive cone has a g.l.b. if and only if for each $x \in P$, the closed interval $[e, x]$ be compact in the interval topology of P.*

[PROOF] The sufficiency is evident from ([4] Theorem 1). Let $\{ x_\alpha : \alpha \in A \}$ and $\{ x_\beta : \beta \in B \}$ are subset of $[e, x]$ such that

$$F = \{ M(x_\alpha) : \alpha \in A \} \cup \{ L(x_\beta) : \beta \in B \}.$$

is a non-void collection with finite intersection property. If B is void set, then $x \in \bigcap F$. If B is not void, we have $x_\alpha \leqq x_\beta$ for each $\alpha \in A$ and $\beta \in B$ by the finite intersection property. And since $\{ x_\beta : \beta \in B \}$ is a non-void bounded, it has a g.l.b. x_0 Then $x_\alpha \leqq x_0$ for each $\alpha \in A$. Clearly, $x_0 \in \bigcap F$.

[LEMMA 4] *The closed interval $[e, x]$ is compact in the interval topology of L if and only if it is compact in the interval topology of the positive cone P.*

[PROOF] Let $\{ x_\alpha : \alpha \in A \}$ *and* $\{ x_\beta : \beta \in B \}$ are subsets of $[e, x]$. And

$$F = \{ M(x_\alpha) : \alpha \in A \} \cup \{ L(x_\beta) : \beta \in B \}$$

is a non-void collection with finite intersection property such that F has a non-void intersection in L, say $x_0 \in \bigcap F$. Then we have $e \leqq x_\alpha \leqq x_0 \leqq x_\beta$ for each $\alpha \in A$ and $\beta \in B$. Therefore, $x_0 \in L(x_\beta) \cap P$ for each $\beta \in B$.

The sufficiency is immediate.

Hence we have

[THEOREM 3] *For l-group L to be complete, it is necessary and sufficient that, for each $x \in P$, $[e, x]$ be compact in the interval topology.*

§ 3. Natural Mapping.

By an l-ideal of a l-group L is meant a normal subgroup of L which contains with any a and b, also all x with $a \cap b \leqq x \leqq a \cup b$.

Already, we know that the factor l-group L/H forms also a l-group if one defines $Hx \cup Hy = Hx \cup y$, $Hx \cap Hy = Hx \cap y$ and $Hx \cdot Hy = Hx \cdot y$, and the natural mapping from L to L/H is a lattice homomorphism,

where, Hx is a residue class containing x *of* L/H.

[LEMMA 5] *Let L be a l-group and H be an l-ideal. Then $M(Hx)=H(M(x))$, $L(Hx)=H(L(x))$ in the factor l-group L/H.*

Where, $M(Hx)=\{Hy : Hx\leqq Hy \text{ in } L/H\}$, $H(Mx)=\{Hy : y\in M(x)\}$

$$L(Hx)=\{Hy : Hx\geqq Hy\},\quad H(L(x))=\{Hy : y\in L(x)\}$$

[PROOF] If $Hx\leqq Hy$, then $Hx\cap y=Hx\cap Hy=Hx$. Therefore, $x\cap y=h\cdot x$ for some $h\in H$, i.e., $x\leqq h^{-1}y$, hence $h^{-1}y\in M(x)$, i. e., $Hy\in H\ M(x)$). Conversely, if $Hy\in H(M(x))$, we have $Hx\leqq Hy$, i.e., $Hy\in M(Hx)$, since $x\leqq y$ and the natural mapping is a lattice homomorphism. And dually.

[THEOREM 4] *The natural mapping from l-group L to factor l-group L/H is a closed mapping in their interval topologies.*

[PROOF] Let S be any closed subset of L in interval topology. It must be the union of a finite number of closed subbasis, hence there are two finite subsets A, B in L such that

$$S=(\cup_{x\in A}M(x))\cup(\cup_{y\in B}L(y)).$$

$$\begin{aligned} H(S)&=H[(\cup_{x\in A}M(x))\cup(\cup_{y\in B}L(y))]\\ &=[\cup_{x\in A}H(M(x))]\cup[\cup_{y\in B}H(L(y))]\\ &=[\cup_{x\in A}M(Hx)]\cup[\cup_{y\in B}L(Hy)] \quad \text{by Lemma 5.}\end{aligned}$$

Therefore, $H(S)$ is a closed subset in interval topology of L/H.

Oct. 7. 1958
Mathematical Department
Liberal Arts and Science College
Kyungpook University

REFERENCES

[1] G. Birkhoff, Lattice Theory, rev. ed. Now York, (1948)

[2] E. S. Northam, The Interval Topology of a Lattice, Proc. Amer. Math. Soc. 4(1953)

[3] R. M. Bare, A Characterization Theorem for Lattice with Hausdorff Interval Topology, Jour. Math. Soc. Japan. 7(1955)

[4] L. E. Wart Jr. , Completeness in Semi-lattice, Can. Jcur. Math. IX (1957)

[5] T. H. Choe, Notes on the Lattice Ordered Group, Kyungpook Math. Jour. 1(1958).