NOTE ON WEAK TOPOLOGY OF BANACH SPACE

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Let X is a Banach space and X* is a conjugate space of X. Then X* is a Banach space, too. The Banach space X is to be weakly compact if and only if there is at least one sequence $\{x_n\}$ of points in S, converging to a point x of X, for an arbitrary infnite subset S of X. [1]

And in Banach space the weakly compactness has the same meaning with compactness for weak topology in Banach space. [2]

Now we have some definitions on weak topological space X*. (The weak topology of X* is the weak topology as functionals in X*.) X* has weak ε-net it means there exist finite points $f_1, f_2, \cdots\cdots, f_n$ of X* such that for each $f \in$ X* there is at least one f_i satisfying the following:

$$|f(x)-f_i(x)|<\varepsilon \qquad \text{for all } x \in X.$$

X* is separable for weak topology if and only if there is a countable subset S* of X and closure of S* for weak topology is X*.

PROPOSITION 1. *X* is weakly compact then X* has ε-net for arbitrary* $\varepsilon>0$.

Proof. If on the contrary there is same $\varepsilon>0$ for which X* has not weak ε-net. So that there is an infinite points sequence $\{f_n\}$ $(n=1, 2, \cdots\cdots)$ of X*, and satisfies.

(1) $$|f_i(x)-f_j(x)|>\varepsilon \quad (i \neq j), \text{ for some } x \in X.$$

$\{f_n\}$ is an infinite points set of X* and X* is weakly compact then there is an element g of X such that g_m converges to g weakly as functional, and $\{g_m\}$ is subsequence of $\{f_n\}$.

Then for arbitrary $\delta>0$ there is an integer n_o such that

$$|g(x)-g_m(x)|<\delta \qquad \text{for all } m>n_o,\ x \in X$$

In the sequence of real numbers, $g_m(x)$ converges to $g(x)$ implies $\{g_m(x)\}$ $(m>n_o)$ is a Cauchy sequence. So we have the following:

for $\varepsilon>0$, $m>n_o$, $l>n_o$

$$|g_m(x)-g_l(x)|<\varepsilon \qquad \text{for all } x \in X.$$

Since $\{g_m\}$ was subsequence of $\{f_n\}$ then we can find out $f_i, f_j \in \{f_n\}$ such that $f_i = g_m, f_j = g_l$, and

(2) $$|f_i(x) - f_j(x)| < \varepsilon.$$

(2) is contrary to (1). Hence for arbitrary $\varepsilon > 0$ X^* has weak ε-net.

PROPOSITION 2. *X^* is weakly compact then X^* is separable for weak topology.*

Proof. From proposition 1, there is at least one weak ε-net correstonding to each $\varepsilon = \frac{1}{2^k}$ $(k=1, 2, 3, \cdots\cdots)$. Let $\{f_n\}$ is the totality of these weak ε-nets. $\{f_n\}$ is a countable subset of X^*. And for arbitrary $f \in X^*$ there are f_{nk} $(k=1, 2, 3, \cdots\cdots)$ such that

$$|f(x) - f_{nk}(x)| < \frac{1}{2^k} \text{ for all } x \in X.$$

So that

$$\lim_{k\to\infty} |f(x) - f_{nk}(x)| = 0 \text{ for all } x \in X.$$

Then f_{nk} converges to f weakly as functional, if k appraoches to ∞.[1] Hence X^* is separable for weak topology.

Banach space X is separable for strong topology then the unit circle of X^* is weakly compact. Refering proposition 2, X is separable for strong topology then the unit circle of X^* is separable for weak topology. We have the other condition X^* to be separable for weak topology. From now in this paper, we assume that X^* is a Banach lattice. [3]

P^* is the totality of $f \in X^*$ such that $f \geqq 0$, where the order of X^* is defined by following formula:

(3) $$f < g \longleftrightarrow f(x) < g(x) \text{ for all } x \in X.$$

PROPOSTION 3. *P^* is separable for weak topology then X^* is separable for weak topology of X^*.*

Proof. Since P^* was separable for weak topology, there is a countable subset Q^* of P^* such that for arbitrary $f \in P^*$ there exists a subsequence $\{f_n\}$ of Q^*:

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in X,$$

1) We dente this term by $w\text{-}\lim_{k\to\infty} f_{nk} = f$.

where ε is arbitrary positive number, $n>n_o$, and n_o is an integer depending on ε.

Each $f\in X^*$ be represented uniquely as

$$f=f^++f^-=f^+-(-f^-),$$

where

$$f^+=f\cap 0, \quad f^-=f\cap 0.$$

Then $f^+\geqq 0$, $f^-\leqq 0$ and $-f^-\geqq 0$. So that f^+, $-f^-$ belong to P^*. And there is a subsequence $\{f_n\}$ of Q^* such that

$$\omega\text{-}\lim_{n\to\infty} f_n=f^+,$$

and a subsequence $\{g_m\}$ of Q^* such that

$$\omega\text{-}\lim_{m\to\infty} g_m=-f^-.$$

$\{f_n-g_m\}$ is a countable subset of X^*, so we can put

$$\{f_n-g_m\}=\{h_i\}$$

by suitable rearrangement. For arbitrary $\varepsilon>0$, there is a number N such that for all m, $n>N$

$$\begin{aligned}&|f_n(x)-g_m(x)-f^+(x)-(-f^-(x)|\\ &\leqq|f_n(x)-f^+(x)|+|-f^-(x)-g_m(x)|\\ &<2\varepsilon \qquad\qquad \text{for all } x\in X.\end{aligned}$$

There is some number M : for arbitrary $i>M$

$$|h_i(x)-f(x)|=|h_i(x)-f^+(x)-(-f^-(y))|<2\varepsilon$$
$$\text{for all } x\in X.$$

Hence

$$\omega-\lim_{i\to\infty} h_i=f.$$

$\{h_i\}$ is a subsequence of S^*:

$$S^*=E\{f+g;\ f\in P^*,\ g\in -P^*\},$$

where $\quad -P=E\{-f\ ;\ f\in P^*\}.$

Then S^* is a countable subset of X^* and X^* is separated by S^* for weak topology of X^*.

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