A CORRESPONDENCE ON LAGUERRE SPACES

By Jae Koo Ahn

1. Introduction.

We consider the special one parameter families of hyperspheres in two Laguerre spaces L^n and $\bar{L}^n$, which are invariant the hyperspheres $V^\lambda+p\overset{(1)}{V}{}^\lambda$ and $\bar{V}^\lambda+p\overset{(1)}{\bar{V}}{}^\lambda$ * respectively. In this paper we shall try to calculate the condition that these families of hyperspheres correspond to each other. The differential geometric properties on this Laguerre space are studied by K. Tomonaga [1]; for this purpose we shall study them.

As the ground of this paper, we begin with the following theorem by K. Tomonaga:

THEOREM *The necessary and sufficient condition that a hypersphere of the form $V^\lambda+p\overset{(1)}{V}{}^\lambda$ is fixed by our connection along the curve $x^i=x^i(t)$, is*

$$\text{(1)}\qquad \frac{\delta \overset{(1)}{V}{}^\lambda}{ds}=0,\qquad \overset{(1)}{V}{}^\lambda=\frac{\delta V^\lambda}{ds}+\frac{dx^\lambda}{ds}.$$

Form this theorem, the differential equation of special one parameter families of hyperspheres which are considering expressed by (1).

2. The correspondence of parameter families (1) of hyperspheres which is invariant between two Laguerre spaces.

Under the assumption $\delta V^\lambda/ds=const.$, we shall try to calculate the condition that these families formed by (1) correspond to each other, and discuss this assumption on 3.

The differential equation (1) in L^n under this assumption formed by

$$\text{(2)}\qquad \frac{d^2x^\lambda}{ds^2}+\Gamma^\lambda_{\mu j}V^\mu{}_{;k}\frac{dx^j}{ds}\frac{dx^k}{ds}+\Gamma^\lambda_{\mu j}\frac{dx^\mu}{ds}\frac{dx^j}{ds}=0.$$

Taking $d\bar{s}^2$ as metric in $\bar{L}^n$ and correspondence $s=s(t)$, $\bar{s}=\bar{s}(t)$ for common parameter t, since $dx^\lambda=0$ $(\lambda=0)$ [1], we can obtain the

*Greek indices take the values 0, 1,……, n and Latin 1,……, n.

following equation as our correspondence

$$(\bar{\alpha}-\alpha)\frac{dx^\sigma}{dt}+(\bar{\Gamma}^\sigma_{\mu j}\bar{V}^\mu{}_{;k}-\Gamma^\sigma_{\mu j}V^\mu{}_{;k})\frac{dx^j}{dt}\frac{dx^k}{dt}$$
$$+(\bar{\Gamma}^\sigma_{jk}-\Gamma^\sigma_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt}=0.$$

Multiplying it by $\frac{dx^\tau}{dt}$, interchanging σ and τ and subtracting these, we obtain

$$(3)\qquad \left(A^\sigma_{jk}\frac{dx^\tau}{dt}-A^\tau_{jk}\frac{dx^\sigma}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}$$
$$+\left(B^\sigma_{jk}\frac{dx^\tau}{dt}-B^\tau_{jk}\frac{dx^\sigma}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}=0.$$

In order that our correspondence may exist, this eqation must be satisfied identically, where

$$(4)\qquad \begin{cases} A^\sigma_{jk}=\bar{\Gamma}^\sigma_{\mu j}\bar{V}^\sigma{}_{;k}-\Gamma^\lambda_{\mu j}V^\mu{}_{;k} \\ B^\sigma_{ik}=\bar{\Gamma}^\sigma_{jk}-\Gamma^\sigma_{jk} \end{cases}$$

And (3) is reduced again to

$$\{(\delta^\tau_i A^\sigma_{jk}-\delta^\sigma_i A^\tau_{jk})+(\delta^\tau_i B^\sigma_{jk}-\delta^\sigma_i B^\tau_{jk})\}\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{dx^k}{dt}=0.$$

Since this must be satisfied identically for all indices, it holds

$$\delta^\tau_i A^\sigma_{jk}+\delta^\tau_j A^\sigma_{ki}+\delta^\tau_k A^\sigma_{ij}+\delta^\tau_k A^\sigma_{ji}+\delta^\tau_i A^\sigma_{kj}+\delta^\tau_j A^\sigma_{ik}$$
$$+\delta^\tau_i B^\sigma_{jk}+\delta^\tau_j B^\sigma_{ki}+\delta^\tau_k B^\sigma_{ij}+\delta^\tau_k B^\sigma_{ji}+\delta^\tau_i B^\sigma_{kj}+\delta^\tau_j B^\sigma_{ik}$$
$$=\delta^\sigma_i A^\tau_{jk}+\delta^\sigma_j A^\tau_{ki}+\delta^\sigma_k A^\tau_{ij}+\delta^\sigma_k A^\tau_{ji}+\delta^\sigma_i A^\tau_{kj}+\delta^\sigma_j A^\tau_{ik}$$
$$+\delta^\sigma_i B^\tau_{jk}+\delta^\sigma_j B^\tau_{ki}+\delta^\sigma_k B^\tau_{ij}+\delta^\sigma_k B^\tau_{ji}+\delta^\sigma_i B^\tau_{kj}+\delta^\sigma_j B^\tau_{ik}.$$

Putting $\tau=i=a$ and contracting, it is reduced to

$$(5)\qquad A^\sigma_{(jk)}+B^\sigma_{jk}=\frac{1}{n+2}\delta^\sigma_a(A^a_{(jk)}+B^a_{jk})+\delta^\sigma_j\varphi_k+\delta^\sigma_k\varphi_j,$$

where $A^\sigma_{(jk)}$ is symmetric part of A^σ_{jk} and

$$\varphi_j=\frac{1}{2(n+2)}(A^a_{ja}+A^a_{aj}+2B^a_{aj}).$$

Writting down fully the components of (5), we have

$$(6)\qquad \begin{cases} A^i_{(jk)}+B^i_{jk}=\frac{n+2}{n+1}(\delta^i_j\varphi_k+\delta^i_k\varphi_j), \\ A^o_{(jk)}+B^o_{jk}=0 \end{cases}$$

Conversely, in cases of $\sigma=\tau=0$; $\sigma=0$, $\tau=h$; $\sigma=h$, $\tau=0$ on (3), it is satisfied clearly. In case of $\sigma=l$, $\tau=m$, adjusting for B^i_{jk} from (6) and substituting it in (3), it is satisfied identically. Hence we have the following theorem:

THEOREM *On two Laguerre spaces which form $\delta V^\lambda/ds=const.$ and $\delta \bar{V}^\lambda/d\bar{s}=const.$, the necessary and sufficient condition that two one parameter families of hyperspheres fixed $V^\lambda+p\overset{(1)}{V^\lambda}$ and $\bar{V}^\lambda+p\overset{(1)}{\bar{V}^\lambda}$ in L^n and $\bar{L}^n$ respectively are correspondent, is that it holds* (4).

3. The Laguerre space with forms $\delta V^\lambda/ds=const.$

We have studied in 2 under the assumption $\delta V^\lambda/ds=const.$ Now we discuss for this assumption.

Let us put

$$\text{(7)}\qquad \frac{\delta V^\lambda}{ds}=c^\lambda,\ \text{i.e.,}\ \frac{dV^\lambda}{ds}+\Gamma^\lambda_{\mu k}V^\mu\frac{dx^k}{ds}=c^\lambda$$

Writting down fully the components of this, we can get

$$\text{(8)}\qquad \frac{dV^i}{ds}+\Gamma^i_{jk}V^j\frac{dx^k}{ds}+\Gamma^i_{ok}V^o\frac{dx^k}{ds}=c^i,$$

$$\frac{dV^o}{ds}+\Gamma^o_{jk}V^j\frac{dx^k}{ds}=c^o.$$

Now, since $\Gamma^o_{jk}=g_{jl}\Gamma^l_{ok}$ [1], the second form of (8) is reduced to

$$\text{(8')}\qquad \frac{dV^o}{ds}+\Gamma^j_{ok}V_j\frac{dx^k}{ds}=c^o\qquad (V_\mu=g_{\lambda\mu}V^\lambda).$$

Contracting the first of (8) by V_i, multiplying (8') by V^o and subtracting these, we have

$$V_i\frac{dV^i}{ds}-V^o\frac{dV^o}{ds}+\Gamma^i_{jk}V^jV_i\frac{dx^k}{ds}=c^iV_i-c^oV^o.$$

Since $V^o=-V_o$ from $V_\lambda=g_{\lambda\mu}V^\mu$, this is reduced to

$$V_\lambda\frac{dV^\lambda}{ds}+\Gamma^i_{jk}V^jV_i\frac{dx^k}{ds}=c^\lambda V_\lambda,$$

and consequently,

$$\text{(9)}\qquad ds=\frac{1}{c^\lambda V_\lambda}\left(V_\lambda\frac{\partial V^\lambda}{\partial x^k}+\Gamma^l_{mk}V_lV^m\right)dx^k.$$

From $ds^2=\bar{g}_{jk}dx^jdx^k$, we have

$$\text{(10)}\qquad \bar{g}_{jk}=\frac{1}{(c^\lambda V_\lambda)^2}\Big\{V_\lambda V_\mu\frac{\partial V^\lambda}{\partial x^j}\frac{\partial V^\mu}{\partial x^k}$$

$$+V_\mu\left(\frac{\partial V^\mu}{\partial x^j}\Gamma^l_{mk}+\frac{\partial V^\mu}{\partial x^k}\Gamma^l_{mj}\right)V_lV^m+\Gamma^a_{bj}\Gamma^c_{dk}V_aV^bV_cV^d\Big\}.$$

Then, Laguerre space which forms $\delta V^{\lambda}/ds$ $const.$ is the space whose foundational Riemannian space is expressed by (9) as its metric tensor.

May 20, 1958
Department of Mathematics,
Kyungpook University.

REFERENCES

[1] K. Tomonaga: A generalization of Laguerre geometry I and II. Journ. of the Math. Soci. of Japan, vol. 2, Nos. 3—4 and vol. 3, No. 2, 1951.

[2] K. Tomonaga: Connections on Laguerre geometry. Sugaku. Math. Soci. of Japan, vol. 2, No. 4.