MODELING ACCURATE INTEREST IN CASH FLOWS OF CONSTRUCTION PROJECTS TOWARD IMPROVED FORECASTING OF COST OF CAPITAL

Gunnar Lucko, Ph.D.[1], Richard C. Thompson, Jr., Ph.D.[2]

[1] Associate Professor of Civil Engineering and Director of Construction Engineering and Management Program, Dept. of Civil Engineering, Catholic University of America, Washington, DC 20064

[2] Post-Doctoral Research Associate, Dept. of Civil Engineering, Catholic University of America, Washington, DC 20064
Correspond to lucko@cua.edu

ABSTRACT: Construction contactors must continuously seek to improve their cash flows, which reside at the heart of their financial success. They require careful planning, analysis, and optimization to avoid the risk of bankruptcy, remain profitable, and secure long-term growth. Sources of cash include bank loans and retained earnings, which are conceptually similar in that they both incur a cost of capital. Financial management therefore requires accurate yet customizable modeling capabilities that can quantify all expenses, including said cost of capital. However, currently existing cash flow models in construction engineering and management have strongly simplified the manner in which interest is assessed, which may even lead to overstating it at a disadvantage to contractors. The variable nature of cash balances, especially in the early phases of construction projects, contribute to this challenging issue.

This research therefore extends a new cash flow model with an accurate interest calculation. It utilizes singularity functions, so called because of their ability to flexibly model changes across any number of different ranges. The interest function is continuous for activity costs of any duration and allows the realistic case that activities may begin between integer time periods, which are often calendar months. Such fractional interest calculation has hitherto been lacking from the literature. It also provides insights into the self-referential behavior of compound interest for variable cash balances. The contribution of this study is twofold; augmenting the corpus of financial analysis theory with a new interest formula, whose strengths include its generic nature and that it can be evaluated at any fractional value of time, and providing construction managers with a tool to help improve and fine-tune the financial performance of their projects.

Keywords: Project management; Cash flows; Singularity functions; Interest; Cost of capital

1. INTRODUCTION

Construction contractors provide professional services in a competitive market with the goal of being profitable. They must therefore plan and continuously control how each project, and indeed each portion thereof, contributes to the overall financial performance of their company.

The remainder of this paper is organized as follows: It reviews the importance of modeling cash flows, describes interest approximations that were used by prior studies, defines singularity functions and provides principles for their use, outlines how they can model cash outflows and inflows, derives exact interest for variable balances, and applies them to an example project from the literature.

2. IMPORTANCE OF CASH FLOWS

Cash flows are extremely important for companies – cash inflows are the financial 'fuel' to remain successful in the business environment, whereas cash outflows are the respective consumption. As in this analogy, shortfalls can endanger the survival of the enterprise [1], even if its productive operations, in this case creating build facilities, are performed as planned and at perfect quality. Means of bridging any temporary gaps between cash outflows and inflows are therefore a fundamental component of sound financial management, specifically obtaining financing, either from capital that is held by the company itself, or by obtaining funds from commercial banks. In both cases, financing fees are incurred from foregoing an opportunity for profitable investment in the former case or from being charged interest by the bank in the latter. In this paper, the differences between these scenarios will not be examined further; rather, the central question is raised of how to properly model such interest in cash flow calculations for construction projects to gain an accurate yet also flexible expression. Prior approaches fell short of this goal. It is necessary to review their assumptions and limitations on interest, develop new equations for interest, and integrate them into a comprehensive model of cash flows over time.

3. INTEREST APPROXIMATIONS

Numerous previous studies that model cash flows have explicitly included interest. However, the discrete nature of their approaches required periodic summations [2] to determine the actual balances at the end of each period when interest is assessed and charged. Moreover, interest is complex in the sense that the behavior of individual elements of a negative balance within periods is variable; activities that cause costs may begin or end at fractional points in time as measured in integer multiples of periods. Furthermore, equations for interest on variable (growing) balances, as opposed to constant amounts borrowed for part of periods or full periods, were completely lacking.

For these reasons, all previous approaches from the literature on construction project management had to approximate interest in one of several different ways [3]:

- *End-of-Period Balance Approximation:* The simplest but also the least desirable and most inaccurate of all approximations relies solely on the balance c_{end} at the end of a period. This calculates financing fees as the interest rate $i \cdot c_{end}$, which grossly overstates its amount for most cash flow scenarios [3]. It effectively assumes that the entire end-of-period balance would have been borrowed since the beginning of the period, whereas a typical behavior is that costs will grow and accumulate throughout a period. It was applied only in few studies, e.g. a recent multi-objective optimization to minimize project duration and maximizing profit [4], a textbook [5], and an older study on initial cash flow estimates [6].
- *Average of Growth Approximation:* This is probably the most commonly used approximation, appearing in numerous studies on cash flow models [7], [8], [9]. It calculates interest into a two-component expression of interest on any balance c_{start} at the start of a period plus interest on the average growth that occurs during said period, i.e. on half of the additional difference (c_{end} - c_{start}). The first component, including any previous financing fees, accrues interest over the entire period. But the second of these components is incorrect insofar as it ignores the concept of *Time Value of Money*. This is captured by Equation 1, whose amount c grows from now to the future at an interest rate i over a duration t.

$$c_{future} = c_{now} \cdot (1+i)^t \qquad (1)$$

Assuming a linear growth from c_{start} to c_{end}, its error lies in the fact that the average additional balance of $\frac{1}{2} \cdot$ ($c_{end} - c_{start}$) is reached after exactly half of the period duration, but *a much larger portion of the actual interest* accrues in the second half of the period rather than in the first half. A simplified calculation shows this effect: Assume that a balance of $100 is borrowed linearly over a period with a duration of one time unit. Splitting this amount into four equal portions of $25, one could approximate the interest as $25 that is borrowed from the first quarter until the period end, $25 from the second quarter to the period end, andsoforth. Taking the midpoint of a quarter as the single point in time where each incremental $25 is borrowed (i.e.

at 0.125, 0.375, 0.625, and 0.875 period), for an assumed interest rate of 5% per period the total interest could be calculated by adding the separate components as follows:
$25 \cdot (1.05^{0.125} - 1) + \$25 \cdot (1.05^{0.375} - 1) + \$25 \cdot (1.05^{0.625} - 1) + \$25 \cdot (1.05^{0.875} - 1) = \$0.1529 + \$0.4616 + \$0.7741 + \$1.0904 = \$2.4790$. In other words, the first half of the period causes only $0.6146 or 24.79% of the interest, but the second half causes $1.8645 or 75.21% of the interest of $2.4790; over three-quarters of the interest is caused by the later borrowing. Compare this with the case that the *Average of Growth Approximation* implies:

Borrowing $50 for the entire period will cause an interest of $50 \cdot (1.05)^1 = \$2.50$, which overcharges the client, in this case the (sub)contractor in favor of the bank. Singularity functions provide the capability to derive a better model for such financing fees in cash flows. Their definition and uses are explained in the following sections.

4. SINGULARITY FUNCTIONS

4.1 Definition of Case Distinction
The definition of singularity functions comprises three elements that must be known to create a valid expression: The intensity of a phenomenon (its strength s), where it begins on the x-axis (its cutoff a), and the nature of its behavior (its exponent n) [10]. The elements are captured in Equation 2, which is the 'basic term' for all models.

$$y(x) = s \cdot \langle x-a \rangle^n = \begin{cases} 0 & if\ x < a\ \ Case\ 1 \\ s \cdot (x-a)^n & if\ x \geq a\ \ Case\ 2 \end{cases} \qquad (2)$$

The case distinction of the pointed bracket operator [11] results in a value of zero for Case 1, meaning that the basic term has not become 'active' yet, and a regularly evaluated function value in Case 2 for all x-values at and to the right of a. The use of the $\geq$ symbol in Case 2 causes Equation 2 to be right-continuous and defined for all x-values. As such, it generalizes the traditional algebraic functions, because they can be 'switched on' only when needed, rather than exhibiting a single behavior across the entire spectrum from minus infinity to plus infinity. This ability makes them ideal to model any complex behaviors of phenomena that change between multiple segments (ranges with starts a_1, a_2, ...) along the x-axis. In the spirit of 'customizing' mathematical expressions to fit a user's needs, it is possible that the simple exponential function within the case distinction provided by Equation 2 could be modified or expanded to more complex expressions.

4.2 Rules for Calculations
Several rules should be followed for the approach of Equation 2 to ensure accurate calculations. Its operator is additively combined into complete singularity functions, i.e. a singularity functions refers to the sum of multiple basic terms. Three principles should be applied after each calculation step to maintain the structure and clarity [12]:

- *Sorting Principle:* Basic terms of the form of Equation 2 should be sorted in a hierarchical order from left to right, for example from lowest to highest a, then from

highest to lowest n, and from highest to lowest s. This is analogous to the standard notation convention for polynomial functions, e.g. $y(x) = 2 \cdot x^2 + 3 \cdot x^1 + 4 \cdot x^0$.

- *Simplification Principle:* Basic terms with the identical cutoff a and exponent n, but different strengths s can and should be simplified by adding their s and writing the expression as a single basic term so that singularity functions when written out are not unnecessarily long.
- *Superposition Principle:* Multiple basic terms that have different a or n capture different aspects of a behavior. In other words, complex behaviors (shapes of $y(x)$) can be modeled by overlaying geometrically simpler shapes, in analogy to discretizing with elements like rectangles or triangles. Note that the number of basic terms that a singularity function may contain is unlimited, so that at least theoretically phenomena of any complexity can be described, even if the computational effort would grow.

The name of singularity functions refers to their unique ability of capturing only those points in time or locations where changes occur, i.e. *singularities* in the otherwise regular behavior of a mathematical phenomenon. This bears a passing resemblance to discrete event simulation, which describes only the events (changes in activity or resource status), and omits any periods in between [13].

For exponent $n = 0$, Equation 2 is a step function with a constant value (where s denotes the step height) from a onward. For $n = 1$, it becomes a slope function with a fixed rate of growth (where s takes on a different meaning of being the slope defined as rise over run) from a onward. Higher exponents would have yet different meanings of s, respectively. It is therefore important to always derive the intended meaning of the strength s from the exponent n.

Importantly, singularities as modeled by the singularity functions can be discontinuities of any exponential order, e.g. sudden vertical steps, horizontal periods of no growth (achieved by subtracting the influence of any terms with $n \geq 1$), or positive or negative changes in slope (bends). In summary, the singularity functions are thus capable to accurately track profiles $y(x)$ of any shape over the x-axis.

4.3 History and New Use in Project Management

First introduced to the project management audience as a new method for analyzing linear schedules [14], [12], singularity functions have actually a long history in the field of structural engineering, where they provided an efficient mathematical approach to calculate shear and moment distributions along structural members [15], [16]. While more well-known by names based on publications by Föppl [17] and Macaulay [18] in the early 1900s, origins can be traced back further to Clebsch [19], [20].

Singularity functions continue to be developed for new applications in construction project management by this author. They include an algorithm to calculate the critical points that compose a continuous critical path through linear schedules based on the sequential activity segments and their buffers [14]; identifying several types of float within the area of the work-time coordinate system of linear schedules [12]; deriving profiles of resource usage directly from the underlying schedule, based on the needs of each activity [21]; and expressing various elements of

cash flows by transforming cost functions into billings and payments in their chronological sequence [22]. The following section provides details on such modeling steps.

Another new application is expressing a recommended amount of cost contingency from an owner's point of view as a singularity function. It decreases as the amount and accuracy of information about the project increases throughout its design and construction phases; uncertainty is reduced particularly once firm bids are received [23].

5. OVERALL CASH FLOW MODEL

4.1 Cost Function

A cost function follows the model of Equation 2 and introduces a unit cost as the slope. This essentially makes the assumption that costs for a specific activity occur in a linearly growing manner or can be approximated as such. This assumption is in alignment with that of most studies on cash flow models, e.g. [24], [25], and [26], but such models were composed of discrete elements, rather than an integrated model as singularity function make possible.

Any such cost has the form of either Equation 3, if it is a single cost c_1 being incurred at a point in time a_1; or Equation 4, if it is a unit cost c_2 that grows throughout a duration d_2 from its start a_{2s} to its finish a_{2f}. Either of them can be tied to activities (direct cost) or occur on its own (indirect cost), e.g. overhead, in which case it is treated as if it is caused by a separate 'office activity' that extends through the duration of the entire project. Note that Equation 4 contains two basic terms [27]: One to add the cost slope c_2 and another to subtract it again. Otherwise, the cost c_2 would incorrectly grow infinitely.

$$y(x)_{cost, fix} = c_1 \cdot \left\langle x - a_1 \right\rangle^0 \qquad (3)$$

$$y(x)_{cost, grow} = \frac{c_2}{a_{2f} - a_{2s}} \cdot \left\langle x - a_{2s} \right\rangle^1 - \frac{c_2}{a_{2f} - a_{2s}}$$

$$\cdot \left\langle x - a_{2f} \right\rangle^1 = \frac{c_2}{d_2} \cdot \left(\left\langle x - a_{2s} \right\rangle^1 - \left\langle x - a_{2f} \right\rangle^1 \right) \qquad (4)$$

4.2 Billing Function

As mentioned, activity costs on construction projects are modeled with Equation 4, which assumes – simplified – that individual purchases of materials, paying wages and benefits, and expenses from either renting or owning equipment occur approximately randomly distributed in time. Further research is planned to refine this working assumption. However, any discrepancy from it will likely remain small, because in the next step, converting costs into bills, all costs that are incurred at any time within a period are accumulated into a single bill at the end of said period. The mathematical operator that accomplishes this is a so-called 'floor' operator that rounds any fractional values down to integers and is written as $\lfloor \ \rfloor$ [28]. When applied not to an individual value but to the x-value of a growing phenomenon $y(x)$, it will convert it into a stepped function. Equation 5 provides its general form, whose only difference to Equation 4 is that its argument x is

always rounded down to integers – only when exceeding the next full integer will the value of the pointed bracket change, giving the desired stepping effect. This assumes that the periods for billing occur at integer points on the time axis, the x-axis. Of course, the bill would not be complete without increasing the cost by the profit margin p and any other adjustments as a factor per Equation 6.

$$y(x)_{step} = \frac{c_2}{d_2} \cdot \left(\left\langle \lfloor x \rfloor - a_{2s} \right\rangle^1 - \left\langle \lfloor x \rfloor - a_{2f} \right\rangle^1 \right) \tag{5}$$

$$y(x)_{bill} = (1+p) \cdot y(x)_{step} \tag{6}$$

4.3 Payment Function

All that remains to convert a billing function into a payment function [22] is determining the delay b that will occur between the two events, and applying any further adjustments to its monetary amount. For the former, a typical assumption is 30 days, or one calendar month [29], but durations can reach even several months, especially for subcontractors [30], [31], as payments are transferred from owner to general contractor, and finally passed on, each of whom has a financial incentive (earning interest) to receive funds quickly but disburse them only slowly.

Equation 7 thus adds a delay b to the start and finish cutoffs a_{2s} and a_{2f}, effectively shifting the entire stepped billing function to the right on the time axis. Of course, any profit p requested by the (sub)contractor and any retainage r withheld by the owner [32] are included as increasing and decreasing factors in the final Equation 8. Further elements at this stage may be individual inflows or outflows from bonuses or penalties, which simply have the already-familiar form of Equation 3, and conditionally depend on certain events happening or not happening. A binary decision variable v can be introduced as a factor to achieve this capability, but is excluded here for brevity.

$$y(x)_{delay} = \frac{c_2}{d_2} \cdot \left(\left\langle \lfloor x \rfloor - a_{2s} - b \right\rangle^1 - \left\langle \lfloor x \rfloor - a_{2f} - b \right\rangle^1 \right) \tag{7}$$

$$y(x)_{payment} = (1-r) \cdot (1+p) \cdot y(x)_{delay} \tag{8}$$

4.4 Financing

Following the *superposition principle*, the cash flows for an overall construction project can now be derived as the sum of all cost functions (cash outflows, Equation 4) plus all payment functions (cash inflows, Equation 8). Any negative balance that occurs at any time throughout the project duration must be financed. Two simplifying assumptions are made here, (1) that the entire cash flow profile can be attributed to a single corporate entity, e.g. one subcontractor, or one contractor, when in reality the overall cash flow is a composite of many smaller ones. And (2) that there exists no difference between interest rates for financing by a loan from a lender (bank) versus using cash holdings of the company (retained earnings) and therefore foregoing earning investment interest [3]. In

other words, the source of temporary cash is considered irrelevant for the purpose of interest analysis in this paper.

6. MODELING INTEREST ACCURATELY

The difference between the sum of all inflows minus the sum of all outflows for any given point in time on the x-axis is the financing need of the specific cash flow profile. In a separate step, financing fees – interest – must be assessed in regular intervals, typically at the end of each period. These fees are individual cash outflows of the type of Equation 3; however, to accurately determine their amount one must consider the exact behavior of cash flows. It is feasible to decompose any overall cash flow profile into the contribution of each individual sequence of costs, billings, and payment. A complicating factor is the fact that activities can start and finish at any fractional point in time [33], not only on integer periods. Progress payments (and bonuses or penalties), on the other hand, are assumed to always occur on integer periods in this initial model. Further research may generalize the timing of all elements further. Interest on a linearly growing balance per Equation 4 must now consider the *Time Value of Money*. Decomposing such a growth into incremental steps is a mathematical series [3] per Equation 9, where the cost within a single period is $c_3 = c_2 / (d_2 \cdot 1 \text{ period})$ and k is the number of increments. Each of its k terms is a small cost stream on which interest accrues, yet each one over a slightly different portion of the period. They are sorted chronologically, from $1/k$-th of the cost that accrues interest over the entire period to $1/k$-th that is borrowed for almost no duration whatsoever. The basic term in Equation 9 indicates that the interest (excluding the balance) is calculated at the end of the first period.

$$y(x)_{int,p,steps} = \frac{c_3}{k} \cdot \left(\left[(1+i)^1 - 1 \right] + ... + \left[(1+i)^{\frac{1}{k}} - 1 \right] \right)$$
$$\cdot \left\langle x - 1 \right\rangle^0 \tag{9}$$

Solving it for the general case requires that k becomes very large. Also, the anticipated fractional start and finish dates for activities must still be implemented. Fortunately, this only reduces the length of the series. If an activity starts at the fractional point in time s during a period, its interest streams are cropped to growing only over $\lceil s \rceil - s$ per Equation 10. On the other hand, if it finishes at a fractional f, its streams will have durations only between 1 and $\lceil f \rceil - f - 1/k$ per Equation 11. Note that the 'ceiling' operator [28] ensures that the interest is assessed at the next integer time, i.e. the end of the period that directly follows a fractional point in time when borrowing ended.

$$y(x)_{int,s,steps} = \frac{c_3}{k} \cdot \left(\left[(1+i)^{\lceil s \rceil - s} - 1 \right] + ... + \left[(1+i)^{\frac{1}{k}} \right. \right.$$
$$\left. \left. - 1 \right] \right) \cdot \left\langle x - \lceil s \rceil \right\rangle^0 \tag{10}$$

$$y(x)_{int,f,steps} = \frac{c_3}{k} \cdot \left(\left[(1+i)^1 - 1 \right] + \ldots + \left[(1+i)^{\lceil f \rceil - f - \frac{1}{k}} - 1 \right] \right) \cdot \left\langle x - \lceil f \rceil \right\rangle^0 \tag{11}$$

Equations 9 through 11 describe how fractional starts, complete periods, and fractional finishes incur interest. Equations 10 and 11 are valid in a single fractional period. To combine them into a single mathematical function, it must be considered that the interest is assessed exactly at an integer end-of-period. For any activity that extends into more than one discrete period, any interest from such integer point in time itself will be compounded, i.e. will cause a further interest stream. All these interest streams, whether from the growing balance itself or compounding interest, are added as the total interest at the last end-of-period, which is the rounded up finish $\lceil f \rceil$ of the activity.

An activity of duration d can have up to three distinct segmental components over time: A fractional start period of $d_s = \lceil s \rceil - s$, one or multiple full periods in its middle, and a fractional finish period of $d_f = f - \lfloor f \rfloor$. The number of full periods is $d_p = d - d_s - d_f$. These definitions are now applied to properly accumulating Equations 9 through 11.

$$y(x)_{int,tot} = \left(y(x)_{int,s} \cdot (1+i)^{\lceil f \rceil - \lceil s \rceil} + y(x)_{int,f} + y(x)_{int,p} \right.$$
$$\left. \cdot \left[(1+i)^{\lceil f \rceil - \lceil s \rceil} + \ldots + (1+i)^{\lceil f \rceil - \lfloor f \rfloor} \right] \right) \cdot \left\langle x - \lceil f \rceil \right\rangle^0 \tag{12}$$

Of course, if $d_p = 1$, then only one full period term is used for its compounding, and if $d_p = 0$ it is omitted. A simplification for Equation 12 is the fact that the series of exponential interest terms as they appear in the full period terms are an *annuity*, a series of periodically compounded payments, which can be calculated with the well-known Equation 13, where A is the repeated annuity payment, n is the number of periods of compounding, and FV is the future value. The only adjustment that must be made is to ensure that the annuity extends from 1 to d_p as is found in Equation 12. Equation 14 can thus be implemented to create the shorter full period term of the new Equation 15.

$$FV = A \cdot \left[(1+i)^{n-1} + \ldots + (1+i)^0 \right] = A \cdot \frac{(1+i)^n - 1}{i} \tag{13}$$

$$FV \cdot (1+i) = A \cdot \left[(1+i)^n + \ldots + (1+i)^1 \right]$$
$$= A \cdot \left(\frac{(1+i)^{n+1}}{i} - 1 \right) \tag{14}$$

$$y(x)_{int,tot} = \left(y(x)_{int,s} \cdot (1+i)^{\lceil f \rceil - \lceil s \rceil} + y(x)_{int,f} + y(x)_{int,p} \right.$$
$$\left. \cdot \left[\frac{(1+i)^{d_p+1}}{i} - 1 \right] \cdot \left\langle d_p - 0 \right\rangle^0 \right) \cdot \left\langle x - \lceil f \rceil \right\rangle^0 \tag{15}$$

To remove the variable k from the three components of Equation 15, i.e. from the underlying Equations 9 through 11, requires making k very large and examining whether or not they then converge to any known expression. It has recently been solved by Lucko and Thompson [3], who let k approach infinity so that the incremental cost c_3/k that causes these many interest streams approaches zero; ultimately, it models a 'triangular' balance of debt within a single period. The solution of such an 'infinitesimal' series of annuities converges to the interest divided by the natural logarithm of one plus the interest per Equation 16.

$$y(x)_{one\ period} = A \cdot \frac{i}{\ln(1+i)} \cdot \left\langle x - 1 \right\rangle^0 \tag{16}$$

Cases of start-fractional and finish-fractional streams of k incremental cost elements are solved analogously, here within a single period, incorporating the aforementioned fractional start or finish periods d_s or d_f, respectively. The length of compounding the incremental streams for start-fractional ranges from d_s to 0 and for finish-fractional it ranges from 1 to 1 - d_f, in both cases measured to their rounded-up integer period when interest is assessed. Shortening the necessary derivation steps of the series development for brevity, the numerators of Equations 17 and 18 are thus curtailed to the difference of exponential terms ranging from d_s to 0 or from 1 to 1 - d_f, respectively. The cutoffs of 1 are used in these generic expressions as a reminder that they only evaluate a single fractional period.

$$y(x)_{fract,start} = A \cdot \frac{(1+i)^{d_s} - (1+i)^0}{\ln(1+i)} \cdot \left\langle x - 1 \right\rangle^0$$
$$= A \cdot \frac{(1+i)^{d_s} - 1}{\ln(1+i)} \cdot \left\langle x - 1 \right\rangle^0 \tag{17}$$

$$y(x)_{fract,finish} = A \cdot \frac{(1+i)^1 - (1+i)^{1-d_f}}{\ln(1+i)} \cdot \left\langle x - 1 \right\rangle^0$$
$$= A \cdot \frac{1+i - (1+i)^{1-d_f}}{\ln(1+i)} \cdot \left\langle x - 1 \right\rangle^0 \tag{18}$$

Compiling these equations into Equation 15 creates the new Equation 19. This comprehensive new expression for an activity with any fractional start and/or finish plus any number of complete periods gives the exact total interest, which is assessed at the integer period $\lceil f \rceil$ that directly follows its finish f. The first term accrues interest from its $\lceil s \rceil$ to $\lceil f \rceil$, the second – for full periods – over a duration that ranges from the number of full periods d_p plus one (a period that may contain another fractional finish) to just one, in analogy to the annuity of Equation 14, the third term accrues interest only over a fractional duration $1 - d_f$. If the second term has a duration of $d_p = 0$, i.e. no full periods, its round bracketed part returns 1, yet applying the pointed bracket operator of singularity functions $\langle \ \rangle$ to d_p itself ensures that the entire second term is set to zero.

$$y(x)_{int,tot} = \frac{c_2}{d_2} \cdot \left(\frac{(1+i)^{d_s} - 1}{\ln(1+i)} \cdot (1+i)^{\lceil f \rceil - \lceil s \rceil} + \frac{i}{\ln(1+i)} \right)$$

$$\cdot \left[\frac{(1+i)^{d_p+1}}{i} - 1 \right] \cdot \langle d_p - 0 \rangle^0 + \frac{1+i-(1+i)^{1-d_f}}{\ln(1+i)} \right)$$

$$\cdot \langle x - \lceil f \rceil \rangle^0 \tag{19}$$

7. EXAMPLE

An example from the literature [5] is used to illustrate the functioning of the new approach of modeling costs and payments that are derived from them with singularity functions, plus the desired accurate interest. Note that the aforementioned billing function is only an intermediate step to derive the final payments, but otherwise does not contribute to the balance behavior of the final cash flows.

The example consists of a small project with four activities, A, B, C, and D. Table 1 provides their start and finish dates in the time unit of periods and their unit costs and total costs. The total project duration is 4.0 periods. This means that with the delay b to receive payments, the final payment is actually not received until 5.0 periods. Additional inputs are \$5,000 per period from overhead as indirect costs, which can be modeled as another activity.

A profit margin of $p = 25\%$ and a retainage of $r = 10\%$ are used; the latter only applied until the project exceeds \$125,000 in constructed value [5], i.e. it is only withheld as an escrow by the owner only in periods one and two, not later. The total project cost to the contractor thus is \$200,000 but its value, including profit, is \$250,000. Note the fractional start of activity C. Interest is 1% per period.

Table 1. Activity List with Costs

Activity	Start	Dur.	Finish	Unit Cost	Total Cost
A	0.0	2.0	2.0	\$ 25,000	\$ 50,000
B	1.0	2.0	3.0	\$ 20,000	\$ 40,000
C	1.5	1.5	3.0	\$ 40,000	\$ 60,000
D	2.0	2.0	4.0	\$ 15,000	\$ 30,000
Overhead	0.0	4.0	4.0	\$ 5,000	\$ 20,000

For brevity, only activity C is modeled with singularity functions in the following sections, all other equations are completely analogous. First, Equation 4 is applied for the costs of activity A in Equation 20. Then it is accumulated in the stepped billing function by applying Equations 5 and 6 to give Equation 21, and finally shifted by the delay per Equations 7 and 8 until the payment is received in Equation 22. Profit and retainage are directly applied to the respective monetary values in the various equations. It is easy at this stage, but omitted for brevity, to include additional variables in the cutoffs of the basic terms of any activity equation [21], e.g. for duration-dependent costs, or if it were modeled that activities may be delayed or shift within their float to different starts and/or finishes.

$$y(x)_{cost,C} = \frac{\$60k}{1.5} \cdot \left(\langle x - 1.5 \rangle^1 - \langle x - 3.0 \rangle^1 \right) \tag{20}$$

$$y(x)_{bill,C} = \frac{\$75k}{1.5} \cdot \left(\langle \lfloor x \rfloor - 1.5 \rangle^1 - \langle \lfloor x \rfloor - 3.0 \rangle^1 \right) \tag{21}$$

$$y(x)_{pay,C} = \frac{\$75k}{1.5} \cdot \left(\langle \lfloor x \rfloor - 2.5 \rangle^1 - \langle \lfloor x \rfloor - 4.0 \rangle^1 \right) \tag{22}$$

Subtracting Equation 22 from Equation 20 gives the individual contribution that C makes to the overall profile of cash flows. Analogous calculations can be made for all other activities of Table 1. Equations 23 and 24 give the cost function and payment function for the entire project. The upper two curves of Figure 1 show these two functions: Costs have the typical S-shape of construction expenses over time; payments have a delayed step-shape.
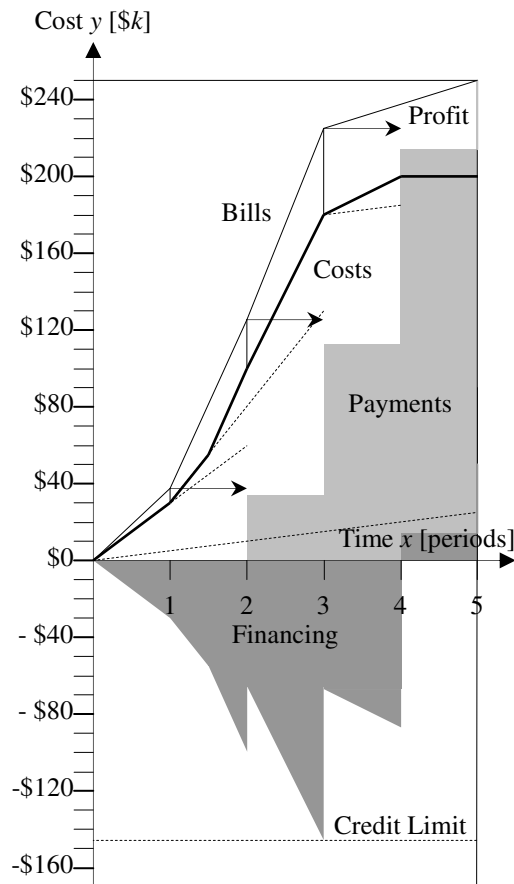

Figure 1. Cash Flow Profile for Example ([5], [27], [34])

Terms of $\langle \lfloor x \rfloor - a \rangle^1$ are equivalent to a series of repeated steps of constant height and are equivalent to arduously modeling multiple separate steps $\langle x - a \rangle^0$ at their cutoffs a. Equation 23 has also been simplified by adding monetary values at identical cutoffs, whose individual contributions are indicated by the indexed activity names. The last term of Equation 24 includes the released retainage, which is 10% of billings that were paid at times $x = 2$ plus $x = 3$, i.e. $0.1 \cdot (\$50k_A + \$20k_B + \$20k_C + 2 \cdot \$5k_{OH}) = \$12.5k$.

$$y(x)_{cost} = \frac{\$30k}{1} \cdot \langle x-0 \rangle^1_{OH,A} + \frac{\$20k}{1} \cdot \langle x-1 \rangle^1_B + \frac{\$40k}{1}$$
$$\cdot \langle x-1.5 \rangle^1_C - \frac{\$30k}{1} \cdot \langle x-2 \rangle^1_{A,B,D} - \frac{\$40k}{1} \cdot \langle x-3 \rangle^1_C$$
$$- \frac{\$20k}{1} \cdot \langle x-4 \rangle^1_{OH,D} \tag{23}$$

$$y(x)_{pay} = \$33.75k \cdot \langle x-2 \rangle^0 + \$78.75k \cdot \langle x-3 \rangle^0$$
$$+ \$100k \cdot \langle x-4 \rangle^1 + \$37.5k \cdot \langle x-5 \rangle^1 \tag{24}$$

Prior research on modeling cash flows with singularity functions was limited by not being able to distinguish several events that all occur at the end-of-period. It was therefore suggested [22] to model the sequence of timings for costs and payments explicitly by delaying the cutoffs a of payments by an incremental duration ε so that cash outflows precede cash inflows at the end of each month, which is a conservative assumption. However, as interest is yet another cash outflow, the proper sequence at the end-of-period should better be as follows: Costs occur at a, interest is added at $a + \varepsilon$, and payments reduce the negative balance at $a + 2 \cdot \varepsilon$. Mathematically the value of ε should be non-zero so that a clear distinction between the three points in time at the end-of-period is possible, yet also small enough that the monetary balances remain the same to the dollars and cent level of accuracy. In this paper the ε and $2 \cdot \varepsilon$ are not shown in the equations for simplicity, yet computer implementations should consider incorporating them to avoid the potential coding problem of equations returning more than one $y(x)$ for any given a.

To finance activity C, the difference between Equations 22 and 20 is examined in detail. As it has a fractional start plus one period, Equation 25 that models how C accrues interest contains components of both Equations 10 and 9. In a sense, interest is self-referential, as it is compounded at the periodic factor of 1.01 once its exact amount has been calculated per the newly derived Equations 16 to 18.

$$y(x)_{int,outflow,C} = - \frac{\$40k}{1} \cdot \left(\frac{1.01^{0.5}-1}{\ln(1.01)} \cdot 1.01^{4-2} \right.$$
$$\left. + \frac{0.01}{\ln(1.01)} \cdot 1.01^1 \right) \cdot \langle x-4 \rangle^0 = -\$20,452.84$$
$$- \$40,601.67 = -\$61,054.51 \tag{25}$$

However, the payments from Equation 22, \$22.5k at $t = 3$ periods (billed \$25k less retainage of \$2.5k) and \$50k at $t = 4$ periods reduces the subsequent balance that earns interest. These constant amounts need to be brought from their respective different cutoffs to the common cutoff a, for the example of C this is selected as receiving its last payment at $t = 4$. Equation 26 therefore compounds all earlier and discounts all later payments, whose amounts are captured by Equation 22 to this common point in time.

$$y(x)_{int,inflow,C} = \left(+ \$22.5k \cdot 1.01^1 + \$50k \cdot 1.01^0 \right.$$
$$\left. + \$2.5k \cdot 1.01^{-1} \right) \cdot \langle x-4 \rangle^0 = + \$22,725 + \$50,000$$
$$+ \$2,475.25 = + \$75,200.25 \tag{25}$$

The value of C at $t = 4$ is - \$61,054.51 + \$75,200.25 = \$14,145.74, or compounded further to the project finish at $t = 5$, its contribution to the overall value of the project is \$14,287.20. Values for the cash flows with exact interest can be obtained analogously for the remaining activities, but are omitted here for brevity. Of course, the simplified nature of this example does not give full justice to the real challenges of cash flow management. Further research is needed to fully adapt its analytical potential to practice.

8. CONCLUSIONS

This paper has presented some of the powerful abilities of singularity functions for modeling financial processes, here the cash inflows and outflows and interest that arises. Its contribution is two-fold, it has derived a flexible new expression to calculate interest exactly by considering the variable balance, rather than resorting to approximations as in prior studies. It also enables integrated modeling of financial phenomena, which can be connected with other aspects of project management, e.g. scheduling and/or resource utilization and their various constraints, and thus can help construction managers in their decision-making to improve the financial performance in a holistic manner.

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REFERENCES

[1] Kenley, R. (1986). *Construction project cash flow modeling.* Thesis submitted to the University of Melbourne in fulfillment of the requirements for the degree of Doctor of Philosophy, Melbourne, Australia: 95 p.

[2] Abido, M. A., Elazouni, A. M. (2011). "Multiobjective evolutionary finance-based scheduling: Entire projects' portfolio." *Journal of Computing in Civil Engineering* 25(1): 85-97.

[3] Lucko, G., Thompson, R. C. (2010). "Derivation and assessment of interest in cash flow calculations for time-cost optimizations in construction project management." *Proceedings of the 2010 42nd Winter Simulation Conference*, eds. Johansson, B., Jain, S., Montoya-Torres, J., Hugan, J., Yücesan, E., Baltimore, Maryland, December 5-8, 2010, Institute of Electrical and Electronics Engineers, Piscataway, New Jersey: 3037-3048.

[4] Senouci, A. B., El-Rayes, K. A. (2009). "Time-profit trade-off analysis for construction projects." *Journal of Construction Engineering and Management* 135(8): 718-725.

[5] Halpin, D. W., Senior, B. A. (2010). *Construction management.* 4th ed., John Wiley and Sons, New York City, New York: 460 p.

[6] Ashley, D. B., Teicholz, P. M. (1977). "Pre-estimate cash flow analysis." *Journal of the Construction Division, Proceedings of the American Society of Civil Engineers* 103(CO3): 369-379.

[7] Elazouni, A. M. (2009). "Heuristic method for multi-project finance-based scheduling." *Construction Management and Economics* 27(2): 199-211.

[8] Liu, S.-S., Wang, C.-J. (2008). "Resource-constrained construction project scheduling model for profit maximization considering cash flow." *Automation in Construction* 17(8): 966-974.

[9] Au, T., Hendrickson, C. (1986). "Profit measures for construction projects." *Journal of Construction Engineering and Management* 112(2): 273-286.

[10] Gross, D., Hauger, W., Schnell, W., Schröder, J., Wall, W. A. (2011). *Technische Mechanik 1: Statik.* [in German: Technical mechanics. Volume 1: Statics.], 11th ed., Springer, Heidelberg, Germany.

[11] Wittrick, W. H. (1965). "A generalization of Macaulay's method with applications in structural mechanics." *American Institute of Aeronautics and Astronautics Journal* 3(2): 326-330.

[12] Lucko, G., Peña Orozco, A. A. (2009). "Float types in linear schedule analysis with singularity functions." *Journal of Construction Engineering and Management* 135(5): 368-377.

[13] Paul, R. J. (1993). "Activity cycle diagrams and the three phase method." *Proceedings of the 25th Winter Simulation Conference*, eds. Evans, G. W., Mollaghasemi, M., Russell, E. C., Biles, W. E., Los Angeles, California, December 12-15, 1993, Institute of Electrical and Electronics Engineers, Piscataway, New Jersey: 123-131.

[14] Lucko, G. (2009). "Productivity scheduling method: Linear schedule analysis with singularity functions." *Journal of Construction Engineering and Management* 135(4): 246-253.

[15] Weissenburger, J. T. (1964). "Integration of discontinuous expressions arising in beam theory." *American Institute of Aeronautics and Astronautics Journal* 2(1): 106-108.

[16] Mahig, J. (1980). "Discontinuity solutions to plate and beam problems." *International Journal of Mechanical Sciences* 6(): 455-460.

[17] Föppl, A. O. (1927). *Vorlesungen über Technische Mechanik. Dritter Band: Festigkeitslehre.* [Lectures on technical mechanics. Volume 3: Solid mechanics.], 10th ed., B. G. Teubner, Leipzig, Germany.

[18] Macaulay, W. H. (1919). "Note on the deflection of beams." *Messenger of Mathematics* 48(9): 129-130.

[19] Chicurel-Uziel, E. (1999). "Non-piecewise representation of discontinuous functions and its application to the Clebsch method for beam deflections." *Meccanica* 34 (4): 281–285.

[20] Clebsch, R. F. A. (1862). *Theorie der Elasticität fester Körper.* [in German: Theory of the elasticity of solid bodies.], B. G. Teubner, Leipzig, Germany.

[21] Lucko, G. (2011). "Integrating efficient resource optimization and linear schedule analysis with singularity functions." *Journal of Construction Engineering and Management* 137(1): 45-55.

[22] Lucko, G. (2011). "Optimizing cash flows for linear schedules modeled with singularity functions by simulated annealing." *Journal of Construction Engineering and Management* 137(7): 523-535.

[23] Zhang, Y., Touran, A. (20120. "A framework for project portfolio management in transit projects." Poster presented at the Construction Research Congress, Purdue University, West Lafayette, Indiana, May 21, 2012, American Society of Civil Engineers, Reston, Virginia.

[24] Alghazi, A., Selim, S. Z., Elazouni, A. M. (2012). "Performance of shuffled frog-leaping algorithm in finance-based scheduling." *Journal of Computing in Civil Engineering* 26(3): 396-408.

[25] Cui, Q., Hastak, M., Halpin, D. W. (2010). "Systems analysis of project cash flow management strategies." *Construction Management and Economics* 28(4): 361-376.

[26] Kenley, R. (2003). *Financing construction: Cash flows and cash farming.* Spon Press, London, United Kingdom: 290 p.

[27] Lucko, G., Thompson, R. C. (2011). "Analysis of cash flow profiles with singularity functions for linear scheduling applications." ed. Plotnick, F. L., *Proceedings of 2011 8th Project Management Institute – Scheduling Community of Practice (formerly College of Scheduling) Annual Conference*, San Francisco, California, May 1-4, 2011, Project Management Institute, Newtown Square, Pennsylvania: 10 p.

[28] Iverson, K. E. (1962). *A programming language.* John Wiley and Sons, New York City, New York.

[29] Dayanand, N., Padman, R. (1997). "On modelling payments in projects." *Journal of the Operational Research Society* 48(9): 906-918.

[30] Elazouni, A. M., Metwally, F. G. (2005). "Finance-based scheduling: Tool to maximize project profit using improved genetic algorithms." *Journal of Construction Engineering and Management* 131(4): 400-412.

[31] Setzer, S. (2009). "Slow payment is sapping contractors' strength: Subs and GCs walk a fine line between getting paid and continuing to work." *Engineering News-Record*, June 24, 2009, 262(21): 20.

[32] Kaka, A. P. (2001). "The case for re-engineering contract payment mechanisms." *Proceedings of the 17th Annual ARCOM Conference*, ed. Akintoye, A., September 5-7, 2001, University of Salford, Association of Researchers in Construction Management 1: 371-379.

[33] Kazaz, B., Sepil, C. (1996). "Project scheduling with discounted cash flows and progress payments." *Journal of the Operational Research Society* 47(10): 1262-1272.

[34] Lucko, G., Cooper, J. P. (2010). "Modeling cash flow profiles with singularity functions." *Proceedings of the 2010 Construction Research Congress*, eds. Ruwanpura, R. Y., Mohamed, Y., Lee, S., Banff, Alberta, Canada, May 8-10, 2010, American Society of Civil Engineers, Reston, Virginia, 2: 1155-1164.