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연속시간 PWM 선형 시스템의 근사 어파인 이산화를 통한 DC-DC 강압 컨버터의 간단한 LQ 준최적 제어 기법

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A Simple LQ Suboptimal Control Scheme for a DC-DC Step-Down Converter Based on Approximate Affine Discretization of Continuous-Time PWM Linear Systems

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Abstract - This paper presents a discrete-time approximate linear model of the continuous-time pulse-width-modulated linear system, and then, by employing the resultant model, a simple LQ suboptimal control scheme is proposed for a DC-DC step-down buck converter. The proposed scheme effectively regulates the output voltage to the desired level, which is also verified by the numerical simulation.

1. Introduction

Pulse-width-modulated (PWM) systems are a class of systems consisting of two subsystems that are alternatively switched depending on the duty ratio and the predetermined switching frequency [1-3]. This kind of systems is of practical importance in the fields of power electronics since almost all of the switched-mode power converters (SMPCs) are intrinsically PWM switched systems [1-4].

Among the DC-DC SMPCs, step-down buck converter is the basic DC-DC converter widely used in many fields of industries. In addition to this applicability, its simple nature motivate the researches on its controller design, and as a result, various control schemes based on the traditional averaged small-signal model [4] and, recently, hybrid model of PWM systems [1-3] were presented. However, averaged small-signal model provides only the local information about the plant, restricting the global design of the controller, and the hybrid model is generally complex and not familiar for the power electronics engineers.

The main contribution of this paper is the development of a simple LQ suboptimal voltage-mode controller for a DC-DC buck converter. The central idea of this work is the approximate affine discretization of a continuous-time (CT) PWM linear system: it converts the CT PWM system into the discrete-time (DT) linear system. The conversion error is normally very small for the well-designed converte, due to the assumption of small-ripple approximation [4], the connection between the power electronics and control theory. For the zero steady state error and LQ suboptimal performance, additional DT state variable is introduced, and then, the control gain is obtained through DT LQR theory [5]. A numerical simulation is performed to verify the effectiveness of the proposed scheme.

2. Discrete-Time Model of Pulse-Width-Modulated System

Consider the following class of PWM single input CT linear system

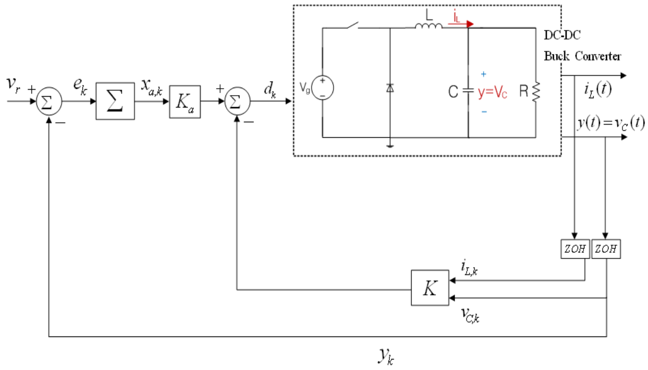
$$\dot{x}(t) = Ax(t) + Bs(t) \tag{1}$$

with the PWM control input $s(t)$:

$$s(t) = \begin{cases} 1, & t \in [\,kT_s,\ (k+d_k)T_s\,) \\ 0, & t \in [\,(k+d_k)T_s,\ (k+1)T_s\,) \end{cases} \tag{2}$$

where $x \in R^n$ is the state of the system; $A \in R^{n \times n}$ and $B \in R^{n \times 1}$ are the system and input coupling matrices, respectively; $T_s > 0$ is the sampling period. The duty ratio d_k is the DT signal satisfying $d_k \in [0,1]$ for any $k \in \{0,1,2,...,\}$. In the sequel, we will exactly discretize (1), and then, derive an approximate DT linear model for sufficiently small $T_s > 0$. For the exact discretization, note that $x((k+d_k)T_s)$ and $x((k+1)T_s)$ can be represented as

$$x((k+d_k)T_s) = e^{Ad_kT_s} \times \left[x(kT_s) + \int_{kT_s}^{(k+d_k)T_s} e^{A(kT_s - \tau)} B\, d\tau \right] \tag{3}$$



<Fig. 1> DC-DC buck converter and its control scheme

$$x((k+1)T_s) = e^{A\bar{d}_k T_s} x((k+d_k)T_s) \tag{4}$$

where $\bar{d}_k := 1 - d_k$. Now, the subtitution of (3) into (4) yields the following DT model with respect to (1):

$$\begin{aligned} x((k+1)T_s) &= e^{AT_s} \times \left[x(kT_s) + \int_{kT_s}^{(k+d_k)T_s} e^{A(kT_s - \tau)} B\, d\tau \right] \\ &= e^{AT_s} x(kT_s) + \int_{kT_s}^{(k+d_k)T_s} e^{A((k+1)T_s - \tau)} B\, d\tau \\ &= e^{AT_s} x(kT_s) + \Phi(d_k) B \end{aligned} \tag{5}$$

where $\Phi(d_k) := \int_0^{d_kT_s} e^{A\tau} d\tau$. Here, the duty ratio d_k is considered as the control input of the system (5), and the PWM control input $s(t)$ is implicitly incorporated into $\Phi(d_k)$. However, $\Phi(d_k)$ in (5) is a nonlinear function of the control input d_k, which makes the design of the controller complicated and difficult. Note that $e^{Ad_kT_s}$ have the Taylor series expansion:

$$e^{Ad_kT_s} = I + Ad_kT_s + \frac{1}{2!} A^2 \times (d_kT_s)^2 + \frac{1}{3!} A^3 \times (d_kT_s)^3 + \cdots$$

which can be approximate by $e^{Ad_kT_s} \approx I + Ad_kT_s$ for sufficiently small $T_s > 0$. By using this fact, $\Phi(d_k)$ can be approximated by a simple term which is linear in d_k.

Lemma 1: Assume that A is invertible. Then, Under the affine approximation $e^{Ad_kT_s} \approx I + Ad_kT_s$, the function $\Phi(d_k)$ is represented by $\Phi(d_k) \approx I \times d_kT_s$.

Proof: The proof is omitted due to space limitation.

By using Lemma 1, one obtains the following approximate DT linear model of the PWM system (1):

$$x_{k+1} = Fx_k + Gd_k, \quad 0 \le d_k \le 1 \tag{6}$$

where $x_k := x(kT_s)$, $F := e^{AT_s}$ and $G := BT_s$. As long as the approximation $e^{Ad_kT_s} \approx I + Ad_kT_s$ is valid, this linear model (6) effectively represents the PWM model (1) almost exactly.

3. Application to DC-DC Step-Down Buck Converter

In this section, the controller design method based on the approximate DT model (6) is presented for the DC-DC step down buck converter given by the following model:

$$A = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix}, \quad B = \begin{bmatrix} V_g/L \\ 0 \end{bmatrix}. \tag{7}$$

with the state $x = [i_L \ \ v_C]^T$ representing the inductor current i_L and the capacitor voltage v_C. The buck converter considered in this paper is depicted in Fig 1. Here, V_g is the input voltage, L and C refer to the capacitance and inductance of the circuit elements, respectively; R represents the load resistance connected at the output of the step-down converter. In this paper, we will focus on the 'voltage mode control', the objective of which is to maintain the output voltage to a desired level. Since the output voltage of the buck converter (7) is the voltage applied to the output capacitor (see Fig. 1), the output equation can be constructed as $y_k = Hx_k$ with $H = [0 \ \ 1]$. Note that y_k represents the output at the DT instant $k \in \{1, 2, 3, \ldots\}$. In the sequel, from $y_k = Hx_k$ and the DT model (6) of the converter (7), the controller for (7) will be designed which have zero-order-hold and updates the duty ratio every DT instant k.

For the design of the controller, consider the approximated DT linear model (6) for the PWM system (1) with (7). For the exact regulation of the output voltage v_C with zero steady-state error in DT domain, an additional state variable should be added as shown in Fig. 1. This variable, denoted by $x_{a,k}$, is defined for the desired output voltage v_r as

$$x_{a,k} = \sum_{i=0}^{k-1} (v_r - Cx_i),$$

which is actually the counterpart of the added state variable in integral control. Note that $x_{a,k+1}$ and $x_{a,k}$ have the following recursive relationship

$$x_{a,k+1} = -Hx_k + x_{a,k} + v_r, \tag{8}$$

Now, by combining (6) with (8), one obtains the following extended DT model:

$$z_{k+1} \equiv \overline{F} z_k + \overline{G} d_k + \overline{G}_r v_r \tag{9}$$

where $z_k := \begin{bmatrix} x_k \\ x_{a,k} \end{bmatrix}$, $\overline{F} := \begin{bmatrix} F & 0 \\ -H & 1 \end{bmatrix}$, $\overline{G} := \begin{bmatrix} G \\ 0 \end{bmatrix}$, $\overline{G}_r := [\,1 \ 1 \ 0\,]^T$.

Here, the only approximated quantity in (9) is $e^{Ad_kT_s}$ shown in Lemma 1. In a well designed converter, $e^{Ad_kT_s} \approx I + Ad_kT_s$ is automatically verified since, to reduce the output switching ripple, the sampling time $T_s > 0$ is designed such that $|T_s| \ll 1/\rho(A)$ holds, where $\rho(X)$ denotes the spectral radius of X. In other words, for the small output ripple, $T_s > 0$ is normally chosen sufficiently small than the intrinsic time constants of the DC-DC converter determined by the circuit elements R, L, C in A. The analysis of a DC-DC converter in power electronics engineering is based on this assumption, commonly known as *small-ripple approximation* [4].

Based on the extended model (9), one can design the discrete-time PWM controller to determine the duty ratio d_k for each period. In this paper, we adopt the linear controller $d_k = -Kx_k - K_a x_{a,k}$ where the control gain (K, K_a) is determined by

$$[\,K \ \ K_a\,] = [\,\overline{G}^T P \overline{G} + R]^{-1} \, \overline{G}^T P \overline{F}. \tag{10}$$

Here, $P \geq 0$ is the solution of the algebraic Riccati equation

$$P = \overline{G}^T P \overline{G} - \overline{F}^T P \overline{G} [\overline{G}^T P \overline{G} + R]^{-1} \overline{G}^T P \overline{F} + Q$$

for the given $Q \geq 0$ and $R > 0$. This is actually the unconstrained optimal control solution minimizing

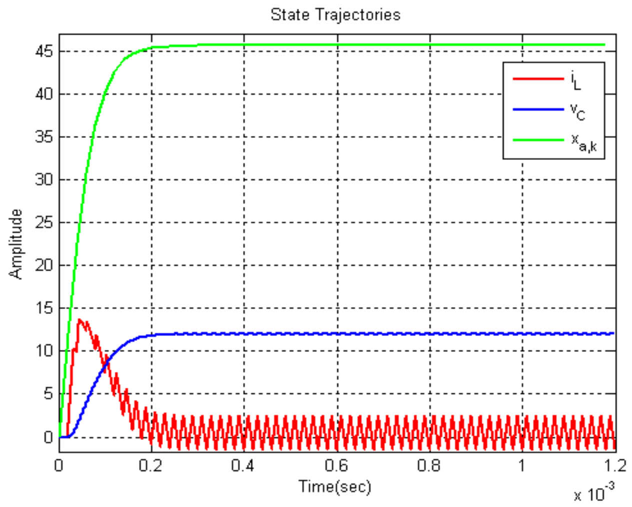
$$J = \sum_{k=0}^{\infty} \left[(z_k - z^*)^T Q (z_k - z^*) + (d_k - d^*) R (d_k - d^*) \right]$$

where z^* and u^* denote the steady state values of z_k and u_k, determined depending on the desired voltage v_r [5]. Since d_k is constrained in $[0, 1]$, one should cautiously choose (Q, R) so that the input-constraint is well-reflected in the value function J.

In order to verify the performance of the proposed controller $d_k = -Kx_k - K_a x_{a,k}$, a numerical simulation is performed with the circuit parameters shown in Table I. The matrices Q and R in the value function J are chosen as $Q = \mathrm{diag}[\,10^{-3} \ 10^{-3} \ 10^2\,]$ and $R = 10^5$, respectively. Fig. 2 demonstrates the state trajectories when (K, K_a) is chosen according to (10). Here, i_L and v_C are CT state variables and $x_{a,k}$ and d_k are the DT state variable and control input, respectively. In Fig 2, all the state variables almost converges within 0.3 [ms] without any oscillatory behavior caused by the proposed controller, showing the good performance of the proposed one.

<Table I> Simulation Parameters

R	L	C	V_g	v_r	$f_s = 1/T_s$
25 [Ω]	30 [μH]	100 [μF]	25 [V]	12 [V]	50 [kHz]



<Fig. 2> State trajectories of the converter control system

4. Conclusions

Based on the exact discretization and Taylor series expansion of the exponentials, this paper presented the approximate DT model which can be easily applied to the well-designed DC-DC step- down buck converter. The theoretical justification of the approximation was also given with the connection to the assumption of the small-ripple approximation. By the additional state variables and the DT LQR theory, a simple LQ suboptimal controller is obtained for voltage regulation. In this procedure, the duty ratio constraint is implicitly incorporated to the control input weights in the value function. and thus, the performance can be further improved when a constrained LQR is considered. This would be the future work of this paper.

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