Proceedings of
The 5th National Congress on Fluids Engineering
March 26-28. 2008, Jeju, Korea

속도차를 갖는 두 회전판에 의해 유도되는 원통 내부 유동

박준상*

Flows in a confined cylindrical container with differential rotating top and bottom disks

Jun Sang Park*

Key Words : *Ekman Layer(에크만 경계층), Rotating Flow(회전유동), Stewartson Layer(스티워슨 경계층)*

Abstract

A theoretical study is made of the flow in a confined cylindrical container with differential rotating top and bottom disks. Two kinds of theoretical solution for the azimuthal velocity were obtained: one is an exact solution of Bessel function type and the other is an approximate solution of exponential function type which comes from WKB approximation. Both theoretical solutions are shown to be self consistent with each other as well as a good agreement with previous studies. Moreover, in a range of relatively low Reynolds number, the obtained solution of Bessel function type shows better result than previous solutions.

1. Introduction

A study is made of the flow in a confined cylindrical container with differential rotating top and bottom disks. The problem to be addressed here is the flow between two finite disks of radius R, the top disk rotating at constant angular speed Ω_T and the bottom disk at Ω_B and the cylindrical sidewall shrouding the disks assumed to rotate at constant angular speed Ω_S. All the above velocities are defined with respect to inertial coordinate. It is assumed that the top and bottom disks, separating at a mean distance H, of which locations are, respectively, defined as $z=H/2$ and $z=-H/2$. For the simplicity, aspect of the container is assumed $R/H=1.0$. Without loss of generality, it assumes that $\Omega_B \geq \Omega_T$ $\Omega_T \leq \Omega_S \leq \Omega_B$ and $(\Omega_B-\Omega_T)/\Omega_S \ll 1$. The system Reynolds number is very large.

This type of problem setting is a basic model of rotating fluid machinery. There are many related previous studies [Batchelor(1951), Stewartson(1957), Lopez(1996)]. It is scarce of theoretical solution investigating the whole flow field in the regions of interior and sidewall boundary layer. Previous theoretical studies focused mainly on local flow characteristics such as inner inviscid region far from the near wall boundary (Batchelor, 1951), sidewall boundary layer (Stewartson, 1957). The global flow characteristics is obtained by a numerical approach (Lopez, 1996).

In this study, by invoking an assumption of Taylor-Proudman column, the governing equation for azimuthal velocity, which is available simultaneously in the interior region and in the sidewall boundary layer, will be built. It will be secured two kinds of theoretical solution : one is an exact solution of governing equation shown as a type of Bessel function and the other is an approximate solution shown as exponential function type. The approximate solution is derived from WKB approximation. Those solutions are a good agreement with previous local studies. The Bessel function type exact solution, which gives better result than previous solutions in a low Reynold number regime, is shown to be very robust solution.

2. The mathematical model

The problem to be addressed here is the flow between two finite disks of radius R^*, the top disk rotating at constant angular speed Ω^*_T and the bottom disk at Ω^*_B and the cylindrical sidewall shrouding the disks assumed to rotate at constant angular speed Ω^*_S. All the above velocities are defined with respect to inertial coordinate. It is assumed that the top and bottom disks, separating at a mean distance H^*, of which locations are, respectively, defined as $z^* = H^*/2$ and $z^* = -H^*/2$. For the simplicity, aspect of the container is assumed $R^*/H^* = 1.0$. Without loss of generality, it assumes that $\Omega^*_B \geq \Omega^*_T$ and $\Omega^*_T \leq \Omega^*_S \leq \Omega^*_B$.

Appropriate non-dimensionalizations are given for velocities and pressure :

$$\vec{V} = \frac{(u^*, v^*, w^*)}{\epsilon \Omega^*_B H^*}, \quad p = \frac{p^* - p^*_0}{\varepsilon \rho^* \Omega^{*2}_B}.$$

The steady governing equations of above problem configuration with respect to rotating coordinate attached at the bottom disk rotating at the angular velocity Ω^*_B, including nonlinear terms, are, in nondimensional form,

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\epsilon\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) - 2v = -\frac{\partial p}{\partial r} + E\left(\nabla^2 u - \frac{u}{r^2}\right), \quad (2)$$

$$\varepsilon\left(u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) + 2u = E\left(\nabla^2 - \frac{v}{r^2}\right), \quad (3)$$

$$\varepsilon\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + E\nabla^2 w, \quad (4)$$

in which $\varepsilon = \dfrac{\Omega^*_B - \Omega^*_T}{\Omega^*_B}$.

And associated boundary conditions are :

$$u = w = 0, v = -r \text{ at } z = \frac{1}{2},$$

$$u = v = w = 0 \text{ at } z = -\frac{1}{2},$$

and $u = w = 0,\ v = -\delta$ at $r = 1$,

in which $\delta = \dfrac{\Omega^*_B - \Omega^*_S}{\Omega^*_B - \Omega^*_T}$. Thus, the value of sidewall angular velocity, δ, ranges $0 \leq \delta \leq 1.0$.

3. Analysis of flow for the interior region far from horizontal disks

It is well known that Taylor-Proudman column prevails in the interior flow out of the horizontal boundary layer, i.e., $\dfrac{\partial u}{\partial z} \approx 0, \dfrac{\partial v}{\partial z} \approx 0$ at $|z| < 1/2 - \Lambda$ where Λ denotes the Ekman layer thickness of $O(E^{1/2})$. Eliminating z-derivative terms from eq.(3), one obtains the governing equation for azimuthal velocity in the interior region as follows

$$\epsilon u_i\left(\frac{\partial v_i}{\partial r} + \frac{v_i}{r}\right) + 2u_i = E\left(\frac{\partial^2 v_i}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_i}{r}\right)\right). \quad (5)$$

It should be emphasized that the above eq. (5) is valid simultaneously in both regions of inner inviscid flow and sidewall boundary layer because the Taylor-Proudman column flow is still sustaining in the extended region. There are many previous investigations that, in the vertical boundary layer near the sidewall, the leading-order azimuthal and radial velocity-fields could be assumed as z-independent function [see Greenspan(1960)]. Thus, in the ensuing discussion, the merged zone consisting of inner inviscid zone and sidewall boundary layer is called as an interior region.

Consider the vertically averaged continuity equation from eq.(1) to obtain the $u - v$ relation, which is needed for solving eq.(5) :

$$\frac{1}{r}\frac{\partial(ru_i)}{\partial r} = -w_i(r, z = 1/2) + w_i(r, z = -1/2). \quad (6)$$

From the consideration of Ekman compatibility condition which has been precisely investigated by many authors [Greenspan(1968)] for a variety of Rossby number ranges, the vertical velocity $w_i(z = \pm 1/2, r)$ in the right-hand-side of eq.(6) becomes, for the weakly nonlinear case, i.e., $\epsilon \ll 1$ as in the below:

for the bottom and top disks, the linear Ekman conditions is expressed as [see, Greenspan(1968); Duck & Forter(2001)] :

$$w_i(z = 1/2) = -\frac{1}{2}E^{1/2}\frac{1}{r}\frac{\partial(rv_i + r^2)}{\partial r}, \quad (7a)$$

$$w_i(z = -1/2) = \frac{1}{2}E^{1/2}\frac{1}{r}\frac{\partial(rv_i)}{\partial r}. \quad (7b)$$

Substituting (7a) & (7b) into (6), the Ekman compatability condition is given as

$$u_i = E^{1/2}\left(v_i + \frac{1}{2}r\right). \quad (8)$$

Combining eq.(5) & (8),

$$\epsilon\left(v_i+\frac{1}{2}r\right)\left(\frac{\partial v_i}{\partial r}+\frac{v_i}{r}\right)+2\left(v_i+\frac{1}{2}r\right)=E^{1/2}\left(\frac{\partial^2 v_i}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_i}{r}\right)\right). \quad (9)$$

After some mathematical manipulations on transforming dependent variable v_i into $\Gamma = r\,v_i$, Eq.(9) becomes

$$\left(\Gamma+\frac{1}{2}\eta\right)\left(1+\epsilon\frac{d\Gamma}{d\eta}\right)=2E^{1/2}\eta\frac{d^2\Gamma}{d\eta^2}, \quad (10)$$

in which $\eta = r^2$.

As assuming the solution of eq.(10) in a series form with an expansion parameter ε as below

$$\Gamma = -\frac{1}{2}\eta + \sum_{n=0}^{\infty}\varepsilon^n\Gamma_n, \quad (11)$$

and substituting eq.(11) into eq.(10), all orders of the equation become :

$$2E^{1/2}\eta\frac{d^2\Gamma_0}{d\eta^2}-\left(1-\frac{1}{2}\epsilon\right)\Gamma_0=0, \quad (12a)$$

$$2E^{1/2}\eta\frac{d^2\Gamma_1}{d\eta^2}-\left(1-\frac{1}{2}\epsilon\right)\Gamma_1=\frac{1}{2}\frac{d\Gamma_0^2}{d\eta}, \quad (12b)$$

$$2E^{1/2}\eta\frac{d^2\Gamma_2}{d\eta^2}-\left(1-\frac{1}{2}\epsilon\right)\Gamma_2=\frac{d(\Gamma_0\Gamma_1)}{d\eta}, \quad (12c)$$

for $n \geq 3$,

$$2E^{1/2}\eta\frac{d^2\Gamma_n}{d\eta^2}-\left(1-\frac{1}{2}\epsilon\right)\Gamma_n=\frac{1}{2}\sum_{i+j=n-1}\frac{d(\Gamma_i\Gamma_j)}{d\eta},$$
$$(i,j=0,1,2,\bullet\bullet\bullet,n-1)$$

and the associated boundary conditions are :

at the rotating axis ($\eta=0$) , i.e., at $r=0$,

$$\Gamma_n(\eta=0)=0\,(n=0,1,2,\cdot\cdot\cdot), \quad (13a)$$

at the sidewall ($\eta=1$), i.e., at $r=1$,

$$\Gamma_0(\eta=1)=-\delta+\frac{1}{2}, \quad (13b)$$

$$\Gamma_n(\eta=1)=0\,(n=1,2,3,\cdot\cdot\cdot). \quad (13c)$$

When $E \ll 1$, the WKB approximation is utilized to obtain the solution to above leading order equation as follows

(1) zero-th order solution ($n=0$)

From the consideration on δ in the boundary condition(13b), we can take two kinds of transformation :

$$\Gamma_0=\exp\left(\Phi_0/\sqrt{2}\,E^{1/4}\right),\ (0\leq\delta\leq 1/2) \quad (14a)$$
$$\Gamma_0=-\exp\left(\Phi_0/\sqrt{2}\,E^{1/4}\right),\ (1/2<\delta\leq 1) \quad (14b)$$

Substituting eq.(14a,b) into eq.(12a), it gives, regardless of δ-value,

$$\sqrt{2}\,E^{1/4}\frac{d^2\Phi_0}{d\eta^2}+\left(\frac{d\Phi_0}{d\eta}\right)^2-\frac{1}{\eta}=0. \quad (15)$$

The boundary conditions are

$$\eta\to 0,\ \Phi_0\to-\infty, \quad (16a)$$

$$\eta=1,\ \Phi_0=\sqrt{2}\,E^{1/4}\ln|-\delta+1/2|. \quad (16b)$$

To obtain the solution of Eq.(15), take a Taylor-series expansion of Φ_0 as expansion parameter $E^{1/4}$

$$\Phi_0=\Phi_0^{(0)}+E^{1/4}\Phi_0^{(1)}+E^{1/2}\Phi_0^{(2)}+\bullet\bullet\bullet. \quad (17)$$

Substituting Eq.(17) into Eq.(15), then , one can obtain each equation (18a)-(18c)to the second order:

$$\left(\frac{d\Phi_0^{(0)}}{d\eta}\right)^2=\frac{1}{\eta}, \quad (18a)$$

$$\frac{d\Phi_0^{(0)}}{d\eta}\frac{d\Phi_0^{(1)}}{d\eta}=-\frac{1}{\sqrt{2}}\frac{d^2\Phi_0^{(0)}}{d\eta^2}, \quad (18b)$$

$$\frac{d\Phi_0^{(0)}}{d\eta}\frac{d\Phi_0^{(2)}}{d\eta}=-\frac{1}{\sqrt{2}}\frac{d^2\Phi_0^{(1)}}{d\eta^2}-\frac{1}{2}\left(\frac{d\Phi_0^{(1)}}{d\eta}\right)^2. \quad (18c)$$

If $E \ll 1$, Γ_0 is obtained by WKB approximation on the above equation

$$\Gamma_0=\left(\frac{1}{2}-\delta\right)\eta^{1/2}\exp\left(\sqrt{2}\,\frac{\eta^{1/2}-1}{E^{1/4}}\right)+O(E^{1/2}). \quad (19)$$

The azimuthal velocity v_i is

$$v_i^{(0)}(r)=-\frac{1}{2}r+\left(\frac{1}{2}-\delta\right)\exp\left(\sqrt{2}\,\frac{r-1}{E^{1/4}}\right)+O(E^{1/2}). \quad (20)$$

It should be noted that the solution (20) is available in the whole region($0\leq r\leq 1$) out of the Ekman boundary layer. Especially, the first term of right-hand-side of Eq.(20) denotes the interior flow solution. The second term shows the Stewartson boundary layer solution which is the same result of previous well known $E^{1/4}$-Stewartson layer studies [(see eq.(3.15a), Heijst(1983)].

In the case of linear problem, an exact solution exists for the zero-th order equation. an exact solution. When $\epsilon \to 0$, the eq.(12a) becomes

$$2E^{1/2}\eta \frac{d^2\Gamma_0}{d\eta^2} - \Gamma_0 = 0. \qquad (21)$$

By using the radial coordinate($r=\sqrt{\eta}$), Eq.(21) is transformed into modified Bessel equation for the leading order azimuthal velocity ($V_0(r) = \Gamma_0/r$).

$$\frac{d^2V_0}{dr^2} + \frac{1}{r}\frac{dV_0}{dr} - \left(2E^{-1/2} + \frac{1}{r^2}\right)V_0 = 0. \qquad (22)$$

The solution of Eq.(22) is

$$V_0(r) = C_1 I_1\left(\frac{\sqrt{2}}{E^{1/4}}r\right) + C_2 K_1\left(\frac{\sqrt{2}}{E^{1/4}}r\right), \qquad (23)$$

where C_1 and C_2 is undetermined constants. From Eq.(13a) and Eq.(13b),

$$C_1 = \left(\frac{1}{2} - \delta\right) / I_1\left(\frac{\sqrt{2}}{E^{1/4}}\right), \qquad C_2 = 0. (24)$$

In summary, the leading order azimuthal velocity $v_i^{(0)}$ is

$$v_i^{(0)} = -\frac{1}{2}r + \left(\frac{1}{2} - \delta\right)\frac{I_1(\sqrt{2}E^{-1/4}r)}{I_1(\sqrt{2}E^{-1/4})}. (25)$$

If $E \ll 1$, i.e., $x\left[\equiv E^{-1/4}r\right] \gg 1$, an asymptotic approximation solution of $I_1(x)$ is

$$I_1(x) \sim \frac{1}{\sqrt{2\pi x}}exp(x) + O\left(\frac{1}{x^{3/2}}\right). \quad (26)$$

Implementing the above formula into Eq.(25), the solution (20) is recovered.

4. Conclusions

A theoretical study is made of the flow in a confined cylindrical container with differential rotating top and bottom disks. The system Reynolds number is very large. By invoking an assumption of Taylor-Proudman column, the governing equation for azimuthal velocity in the interior region was built. Two kinds of theoretical solution for the azimuthal velocity were obtained: one is an exact solution of Bessel function type and the other is an approximate solution of exponential function type which comes from WKB approximation. Both theoretical solutions are shown to be self consistent with each other as well as a good agreement with previous studies. Moreover, in a range of relatively low Reynolds number, the obtained solution of Bessel function type shows better result than previous solutions.

Acknowledgement

"This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD)" (KRF-2007-521-D00061).

References

[1] G.K. Batchelor, Note on a class of solutions of the Navier-Stokes equations representing steady rotationally-symmetric flow, Quart. J. Mech. & Applied Math., 4(1) (1951), pp.29-41.
[2] K. Stewartson, On almost rigid rotations, J. Fluid Mech., 3 (1957), pp.17-26.
[3] J.M. Lopez, Flow between a stationary and a rotating disk shrouded by a co-roating cylinder, Phys. Fluids, 8(10) (1996), pp.2605-2613.
[4] G.J.F. van Heijst, The shear-layer structure in a rotating fluid near a differentially rotating sidewall, J. Fluid Mech., 130 (1983), pp.1-12.
[5] H.P. Greenspan, The theory of rotating fluids, Cambridge Univ. Press (1968)