A STUDY ON STRONGLY REDUCED AND REGULAR NEAR-RINGS

Yong Uk Cho

Department of Mathematics Education, Silla University,Pusan 617-736, Korea, e-mail: yucho@silla.ac.kr

Abstract

A near-ring N is called strongly reduced if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$, where N_c denotes the constant part of N. We investigate some properties of strongly reduced near-rings and apply those to the study of left strongly regular near-rings. Finally we classify some reduced, and strongly reduced near-rings.

Key words :near-rings, strongly reduced near-rings

1. Introduction

A right near-ring N is called left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Mason [3] introduced this notion and characterized left strongly regular zero-symmetric near-rings. Several authors ([2], [4], [5], [7] etc.) studied them. In particular, Reddy and Murty [7] extended some results in [3] to the non-zero symmetric case.

Let N be a right near-ring and let N_c denote the constant part of N. We define N to be strongly reduced if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously a strongly reduced near-ring N is reduced. We show that strong reducedness is a more general concept than the property (*) in Reddy and Murty [7]. Left or right strongly regular near-rings form one of the important class of strongly reduced near-rings. Using strong reducibility, we characterize left strongly regular near-rings and (P_0)-near-rings.

2. Results

Throughout this paper we work with right near-rings. For notation and basic results, we shall refer to Pilz [6]. Recall that a near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. For a near-ring N, N_c denotes the constant part of N, that is, $N_c = \{x \in N \mid x = x0\}$. A near-ring N is said to be *strongly reduced* if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously N is strongly reduced if and only if, for $a \in N$ and any positive integer n, $a^n \in N_c$ implies $a \in N_c$. We will show that a strongly reduced near-ring is reduced, that is, for $a \in N$, $a^2 = 0$ implies $a = 0$.

Proposition 2.1. (1) Let N be a near-ring. If $a \in < a^2 >$ for each $a \in N$, then N is strongly reduced. In particular, right or left strongly regular near-rings are strongly reduced.

(2) Every integral near-ring N is strongly reduced. Hence a subdirect sum of integral near-rings is strongly reduced.

We state some basic properties of a strongly reduced near-ring.

Proposition 2.2. Let N be a strongly reduced near-ring and let $a, b, x \in N$. Then we have the following.

(1) N is reduced.

(2) If $ab^n \in N_c$ for some positive integer n, then $\{ab, ba\} \cup aNb \cup bNa \subseteq N_c$.

(3) If $ab^n = 0$ for some positive integer n, then $ab = 0$ and $ba = b0$.

Clearly, if N is a zero-symmetric near-ring, then N is strongly reduced if and only if N is reduced. Following Reddy and Murty [7] we say that a near-ring N has the property (*) if it satisfies

(i) for any $a, b \in N$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

We give eqivalent conditions for a near-ring N to be strongly reduced.

Theorem 2.3. The following statements are equivalent for a near-ring N:

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(1) N is strongly reduced.
(2) For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.
(3) N has the property (*).
(4) If $a^{n+1} = xa^{n+1}$ for $a, x \in N$ and some nonnegative integer n, then $a = xa = ax$.

The following is a generalization of [7, Theorem 3].

Lemma 2.4. Let N be a strongly reduced near-ring and let $a, x \in N$. If $a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.

A near-ring N is said to be *left π-regular* if, for each $a \in N$, there exists a positive integer n and an element $x \in N$ such that $a^n = xa^{n+1}$. Here we give some characterizations of left strongly regular near-rings.

Theorem 2.5. Let N be a near-ring. Then the following statements are equivalent:
1) N is left strongly regular.
2) N is strongly reduced and left π-regular.
3) For each $a \in N$, there exists $x, y \in N$ such that $a = xa^2ya$.
4) For each $a \in N$, $a \in < a^2 > \cap aNa$.

A near-ring is said to be *periodic* if, for each $a \in N$, there exist distinct positive integers m, n such that $a^m = a^n$. A near-ring N is called a (P_0)-near-ring if, for each $a \in N$, there exists an integer $n > 1$ such that $a = a^n$ (see [6, 9.4, p.289]).

Corollary 2.6. Let N be a near-ring. Then the following statements are equivalent:
1) N is periodic and strongly reduced.
2) N is a (P_0)-near-ring.

References

[1] Clay, J. R., *The near-rings on groups of low order*, Math. Z. **104** (1968), 364–371.

[2] Hongan, M., *Note on strongly regular near-rings*, Proc. Edinburgh Math. Soc. **29** (1986), 379-381.

[3] Mason, G., *Strongly regular near-rings*, Proc. Edinburgh Math. Soc. **23** (1980), 27-35.

[4] Mason, G., *A note on strong forms of regularity for near-rings*, Indian J. of Math. **40(2)** (1998), 149-153.

[5] Murty, C. V. L. N., *Generalized near-fields*, Proc. Edinburgh Math. Soc. **27** (1984), 21-24.

[6] Pilz, G., *Near-Rings*, North-Holland Publishing Company, Amsterdam, New York, Oxford 1983.

[7] Reddy, Y. V. and Murty, C. V. L. N., *On strongly regular near-rings*, Proc. Edinburgh Math. Soc. **27** (1984), 61–64.