

Common fixed point theorem and example in intuitionistic fuzzy metric space

직관적 퍼지 거리공간에서 공통부동점 정리 및 예제

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Abstract

We will establish common fixed point theorem and example for four self maps in intuitionistic fuzzy metric space.

Keywords : Common fixed point theorem, t-norm, t-conorm, intuitionistic fuzzy metric space.

1. Introduction

Recently, Park et.al.[7] defined the intuitionistic fuzzy metric space, and Park et.al.[4] proved a fixed point theorem of Banach for the contractive mapping of a complete intuitionistic fuzzy metric space.

In this paper, we will establish common fixed point theorem for four self maps in intuitionistic fuzzy metric space. These results have been used to obtain translation and generalization of Grabiec's contraction principle.

2. Preliminaries

Definition 2.1. ([3]) A operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -norm if $*$ is satisfying the following conditions:

- (a) $*$ is commutative and associative,
- (b) $*$ is continuous,
- (c) $a * 1 = a$ for all $a \in [0, 1]$,
- (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 2.2. ([3]) A operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -conorm if \diamond is satisfying the following conditions:

- (a) \diamond is commutative and associative,
- (b) \diamond is continuous,
- (c) $a \diamond 0 = a$ for all $a \in [0, 1]$,
- (d) $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 2.3. ([3]) The 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm, \diamond is a continuous t -conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

- (a) $M(x, y, t) > 0$,
- (b) $M(x, y, t) = 1 \iff x = y$,
- (c) $M(x, y, t) = M(y, x, t)$,
- (d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (e) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
- (f) $N(x, y, t) > 0$,
- (g) $N(x, y, t) = 0 \iff x = y$,
- (h) $N(x, y, t) = N(y, x, t)$,
- (i) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
- (j) $N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Note that (M, N) is called an intuitionistic fuzzy metric on X . The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t , respectively.

Definition 2.4. ([6]) Let X be an intuitionistic fuzzy metric space. (a) A sequence $\{x_n\}$ in X is convergent to x in X iff $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$, $\lim_{n \rightarrow \infty} N(x_n, x, t) = 0$ for each $t > 0$. (b) A sequence $\{x_n\}$ in X is called Cauchy sequence iff $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0$ for all $t > 0$ and $p > 0$. (c) X is called complete if every Cauchy sequence is convergent in it.

Definition 2.5. Functions M and N are continuous in intuitionistic fuzzy metric space iff whenever $\{x_n\} \rightarrow$

x and $\{y_n\} \rightarrow y$ then

$$\lim_{n \rightarrow \infty} M(x_n, y_n, t) = M(x, y, t),$$

$$\lim_{n \rightarrow \infty} N(x_n, y_n, t) = N(x, y, t) \text{ for all } t > 0.$$

Lemma 2.6. ([5]) Let X be an intuitionistic fuzzy metric space. If there exists a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$M(x, y, kt) \geq M(x, y, t), \quad N(x, y, kt) \leq N(x, y, t),$$

then $x = y$.

3. Compatible mapping

Definition 3.1. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. The mappings are said to be compatible if

$$\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1,$$

$$\lim_{n \rightarrow \infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x$ for some $x \in X$.

Definition 3.2. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. The mappings are said to be weak-compatible if they commute at their coincidence points. that is, $Ax = Bx$ implies $ABx = BAx$.

Definition 3.3. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. The mappings are said to be semi-compatible if

$$\lim_{n \rightarrow \infty} M(ABx_n, Bx, t) = 1,$$

$$\lim_{n \rightarrow \infty} N(ABx_n, Bx, t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x \in X$.

Lemma 3.4. Let A, B be self mappings on intuitionistic fuzzy metric space X . If B is continuous, then (A, B) is semi-compatible iff (A, B) is compatible.

4. Main Results

Theorem 4.1. Let A, B, S and T be self mappings of a complete intuitionistic fuzzy metric space satisfying

- (a) $A(X) \subset T(X), B(X) \subset S(X)$,
- (b) A or S is continuous,

(c) Pair (A, S) is semi-compatible and (B, T) is weak-compatible,

(d) There exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \min\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\},$$

$$N(Ax, By, kt) \leq \max\{N(By, Ty, t), N(Sx, Ty, t), N(Ax, Sx, t)\},$$

(e) $\lim_{t \rightarrow \infty} M(x, y, t) = 1, \lim_{t \rightarrow \infty} N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$.

Then A, B, S and T have unique common fixed point in X .

Proof. Let $x_0 \in X$ be any arbitrary point as $A(X) \subset T(X)$ and $B(X) \subset S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively construct sequences $\{x_n\}, \{y_n\}$ such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \dots$. Now using (d) with $x = x_{2n}, y = x_{2n+1}$,

$$M(Ax_{2n}, Bx_{2n+1}, kt) \geq \min\{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)\},$$

$$N(Ax_{2n}, Bx_{2n+1}, kt) \leq \max\{N(y_{2n+1}, y_{2n+2}, t), N(y_{2n}, y_{2n+1}, t)\}.$$

Therefore

$$M(y_{2n+1}, y_{2n+2}, kt) \geq \dots \geq \min\{M(y_{2n+1}, y_{2n+2}, \frac{t}{k^m}), M(y_{2n}, y_{2n+1}, t)\},$$

$$N(y_{2n+1}, y_{2n+2}, kt) \leq \dots \leq \max\{N(y_{2n+1}, y_{2n+2}, \frac{t}{k^m}), N(y_{2n}, y_{2n+1}, t)\}.$$

Taking limit as $m \rightarrow \infty$, we get, for all $t > 0$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$$

$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t).$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$$

$$N(y_{2n+2}, y_{2n+3}, kt) \leq N(y_{2n+1}, y_{2n+2}, t).$$

Thus for all n and $t > 0$,

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

$$N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t).$$

Therefore

$$M(y_n, y_{n+1}, t) \geq \dots \geq M(y_0, y_1, \frac{t}{k^n}),$$

$$N(y_n, y_{n+1}, t) \leq \dots \leq N(y_0, y_1, \frac{t}{k^n}).$$

Hence

$$\lim_{n \rightarrow \infty} M(y_n, y_{n+1}, t) = 1,$$

$$\lim_{n \rightarrow \infty} N(y_n, y_{n+1}, t) = 0$$

for all $t > 0$.

Now, for any integer p ,

$$M(y_n, y_{n+p}, t)$$

$$\geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p})$$

$$\dots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}),$$

$$N(y_n, y_{n+p}, t)$$

$$\leq N(y_n, y_{n+1}, \frac{t}{p}) \diamond N(y_{n+1}, y_{n+2}, \frac{t}{p})$$

$$\dots \diamond N(y_{n+p-1}, y_{n+p}, \frac{t}{p}).$$

Therefore

$$\lim_{n \rightarrow \infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \dots * 1 = 1,$$

$$\lim_{n \rightarrow \infty} N(y_n, y_{n+p}, t) \leq 0 \diamond 0 \diamond \dots \diamond 0 = 0.$$

Hence $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore $\{y_n\}$ converges to $z \in X$. Its subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ also converges to z . i.e.,

$$\{Ax_{2n}\} \rightarrow z, \quad \{Bx_{2n+1}\} \rightarrow z$$

$$\{Sx_{2n}\} \rightarrow z, \quad \{Tx_{2n+1}\} \rightarrow z.$$

Let A be continuous function. Then

$$AAx_{2n} \rightarrow Az, \quad ASx_{2n} \rightarrow Az.$$

Since pair (A, S) is semi-compatibility, as limit of a sequence in intuitionistic fuzzy metric space is unique, we have $Az = Sz$.

First, putting $x = z, y = x_{2n+1}$ in (d), we get

$$M(Az, Bx_{2n+1}, kt)$$

$$\geq \min\{M(Bx_{2n+1}, Tx_{2n+1}, t),$$

$$M(Sz, Tx_{2n+1}, t), M(Az, Sz, t)\},$$

$$N(Az, Bx_{2n+1}, kt)$$

$$\leq \max\{N(Bx_{2n+1}, Tx_{2n+1}, t),$$

$$N(Sz, Tx_{2n+1}, t), N(Az, Sz, t)\}.$$

Taking limit as $n \rightarrow \infty$,

$$M(Az, z, kt) \geq M(Az, z, t), \quad N(Az, z, kt) \leq N(Az, z, t).$$

By Lemma 2.6, $Az = z$. Thus $Az = Sz = z$.

Second, since $A(X) \subset T(X)$, there exist $u \in X$ such that $z = Az = Tu$. Putting $x = x_{2n}, y = u$ in (d), we get

$$M(Ax_{2n}, Bu, kt) \geq \min\{M(Bu, Tu, t),$$

$$M(Sx_{2n}, Tu, t), M(Ax_{2n}, Sx_{2n}, t)\},$$

$$N(Ax_{2n}, Bu, kt) \leq \max\{N(Bu, Tu, t),$$

$$N(Sx_{2n}, Tu, t), N(Ax_{2n}, Sx_{2n}, t)\}.$$

Taking limit as $n \rightarrow \infty$, we get

$$M(z, Bu, kt) \geq M(Bu, z, t), \quad N(z, Bu, kt) \leq N(Bu, z, t).$$

for all $t > 0$. By Lemma 2.6, $z = Bu = Tu$ and weak compatibility of (B, T) gives $TBu = BTu$. i.e., $Tz = Bz$.

Third, putting $x = z, y = z$ in (d), we have

$$M(z, Bz, kt) \geq \min\{M(Bz, Tz, t),$$

$$M(Sz, Tz, t), M(Az, Sz, t)\},$$

$$N(z, Bz, kt) \leq \max\{N(Bz, Tz, t),$$

$$N(Sz, Tz, t), N(Az, Sz, t)\}.$$

Using the results from Second, we get

$$M(z, Bz, kt) \geq M(z, Bz, t), \quad N(z, Bz, kt) \leq N(z, Bz, t).$$

for all $t > 0$, which gives $Bz = z$. Hence we get $z = Az = Bz = Sz = Tz$. that is, z is common fixed point of A, B, S and T .

Now, let S be continuous function. Then $Sx_{2n} \rightarrow Sz, S^2x_{2n} \rightarrow Sz$. Since pair (A, S) is semi-compatibility, $\lim_{n \rightarrow \infty} ASx_{2n} = Sz$.

Fourth, putting $x = Sx_{2n}, y = x_{2n+1}$ in (d), we get

$$M(ASx_{2n}, Bx_{2n+1}, kt)$$

$$\geq \min\{M(Bx_{2n+1}, Tx_{2n+1}, t),$$

$$M(SSx_{2n}, Tx_{2n+1}, t), M(ASx_{2n}, SSx_{2n}, t)\},$$

$$N(ASx_{2n}, Bx_{2n+1}, kt)$$

$$\leq \max\{N(Bx_{2n+1}, Tx_{2n+1}, t),$$

$$N(SSx_{2n}, Tx_{2n+1}, t), N(ASx_{2n}, SSx_{2n}, t)\}.$$

Taking limit as $n \rightarrow \infty$,

$$M(Sz, z, kt) \geq M(Sz, z, t), \quad N(Sz, z, kt) \leq N(Sz, z, t).$$

for all $t > 0$. By above Lemma 2.6, $Sz = z$.

Fifth, putting $x = z, y = x_{2n+1}$ in (d),

$$\begin{aligned} &M(Az, Bx_{2n+1}, kt) \\ &\geq \min\{M(Bx_{2n+1}, Tx_{2n+1}, t), \\ &\quad M(Sz, Tx_{2n+1}, t), M(Az, Sz, t)\}, \\ &N(Az, Bx_{2n+1}, kt) \\ &\leq \max\{N(Bx_{2n+1}, Tx_{2n+1}, t), \\ &\quad N(Sz, Tx_{2n+1}, t), N(Az, Sz, t)\}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$,

$$M(Az, z, kt) \geq M(Az, z, t), N(Az, z, kt) \leq N(Az, z, t).$$

By Lemma 2.6, $Az = z$. Therefore $Az = Sz = z$. Applying Second and Third, $Tz = Bz = z$. Thus $z = Az = Bz = Sz = Tz$. That is, z is unique common fixed point of A, B, S and T .

Finally, we show the uniqueness of common fixed point. Let w be another common fixed point of A, B, S and T . Then $w = Aw = Bw = Sw = Tw$. Putting $x = z$ and $y = w$ in (d), therefore

$$M(z, w, kt) \geq M(z, w, t), \quad N(z, w, kt) \leq N(z, w, t).$$

Thus $z = w$. Hence z is common fixed point of the four self maps A, B, S and T . \square

Corollary 4.2. Let A be a self mappings of a complete intuitionistic fuzzy metric space X satisfying $\lim_{t \rightarrow \infty} M(x, y, t) = 1, \lim_{t \rightarrow \infty} N(x, y, t) = 0$ such that, for all $x, y \in X, k \in (0, 1)$ and $t > 0$

$$\begin{aligned} M(Ax, Ay, kt) &\geq M(x, y, t), \\ N(Ax, Ay, kt) &\leq N(x, y, t). \end{aligned}$$

Then A has a unique fixed point in X .

Example 4.3. Let (X, d) be the metric space with $X = [0, 1]$. Denote $a * b = ab$ and $a \diamond b = \min\{1, a+b\}$ for all $a, b \in [0, 1]$ and let M, N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows :

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then (M, N) is an intuitionistic fuzzy metric on X and $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space.

Define self mappings A, B, S and T by

$$\begin{aligned} A(X) &= \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{3} & \text{otherwise} \end{cases}, \\ S(X) &= \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{otherwise} \end{cases}, \\ B(X) &= \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3} & \text{otherwise} \end{cases}, \\ T(X) &= \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{3} & \text{if } 0 < x \leq \frac{1}{4} \\ 1 & \text{if } \frac{1}{4} < x \leq 1. \end{cases} \end{aligned}$$

Then S is continuous, the pair (A, S) is semi-compatible and (B, T) is weak-compatible. Also, $A(X) = B(X) = \{0, \frac{1}{3}\}, S(X) = [0, 1]$ and $T(X) = \{0, \frac{1}{3}, 1\}$ satisfy the containment conditions of Theorem 4.1. For $k = \frac{1}{2}$, (d) of Theorem 4.1 is satisfied and we obtain that 0 is the unique common fixed point of four mappings.

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