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Geometrically nonlinear analysis of thin-walled open-section composite beams

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This paper presents a flexural-torsional analysis of thin-walled open-section composite beams. A general geometrically nonlinear model for thin-walled composite beams and general laminate stacking sequences is given by using systematic variational formulation based on the classical lamination theory. The nonlinear algebraic equations of present theory are linearized and solved by means of an incremental Newton−Raphson method. Based on the analytical model, a displacement-based one-dimensional finite element model is developed to formulate the problem. Numerical results are obtained for thin-walled composite beams under general loadings, addressing the effects of fiber angle, laminate stacking sequence and loading parameters.

Keywords: thin-walled composites, classical lamination theory, flexural-torsional response, nonlinear theory

I. INTRODUCTION

Fiber-reinforced composite materials have been used over the past few decades in a variety of structures. Thin-walled composite structures are often very thin and have complicated material anisotropy. The theory of thin-walled closed section members made of isotropic materials was first developed by Vlasov [1] and Gjelsvik [2]. Bauld and Tzeng [3] presented nonlinear model for thin-walled composites by extending Gjelsvik's formulation to the balanced symmetric laminated composite materials. Gupta et al [4] developed a two-noded, 8 degrees of freedom per node thin-walled open-section laminated anisotropic beam finite element. Chandra and Chopra [5] presented a theoretical-cum-experimental study on the static structural response of composite I-beams with elastic couplings. Bhaskar and Librescu [6] developed non-linear theory of composite thin-walled beams, which were employed in a broad field of engineering problems. Omidvar and Ghorbanpoor [7] developed a nonlinear finite element model for thin-walled open-section structural members made of laminated composites with symmetric stacking sequence. Based on the asymptotic analysis of the classical shell theory, Atilgan and Hodges et al. [8,9] developed geometrically nonlinear behavior of anisotropic beams. By using the computer program so-called "Variational Asymptotic Beam Section Analysis" (VABS), Cesnik, Hodges and Yu et al. [10,11] performed studies of thin-walled composite beams. VABS used the variational asymptotic method (VAM) to split a three-dimensional nonlinear elasticity problem into a two-dimensional linear cross-sectional analysis and a one-dimensional, nonlinear beam problem. Fraternali and Feo [12] developed a small strain and moderate rotation theory of laminated composite thin-walled beams by generalizing the classical Vlasov theory. Special attention, deserve the works of Cortinez and Piovan et al. [13,14,15] who introduced non-linear model for of thin-walled composite beams with shear deformation. This model incorporated, in a full form, the shear flexibility (bending and non-uniform warping), featured in a consistent way by means of a linearized formulation based on the Reissner's Variational Principle. However, it was strictly valid for symmetric balanced laminates and especially orthotropic laminates. Moreover, Piovan and Cortinez [15] also developed a new theoretical model for the dynamic, static and buckling analysis of anisotropic, open and closed cross-section composite thin-walled-beams with general stacking sequences and arbitrary states of initial stresses and off-axis loadings. Recently, a general geometrically nonlinear theory was derived by Lee [16] to study the lateral buckling of thin-walled composite beams with monosymmetric sections.

In the present study, the analytical model developed by Lee [17] is extended by incorporating geometric nonlinearity. A general geometrically nonlinear model for thin-walled composite beams and general laminate stacking sequences is given by using systematic variational formulation based on the classical lamination theory. The nonlinear algebraic equations of present theory are linearized and solved by means of an incremental Newton−Raphson method. Based on the analytical model, a displacement-based one-dimensional finite element model that accounts for the geometric nonlinearity in the von Kármán sense is developed to formulate the problem. Numerical results are obtained for thin-walled composite beams under general external loadings, addressing the effects of fiber angle, laminate stacking sequence and loading parameters.

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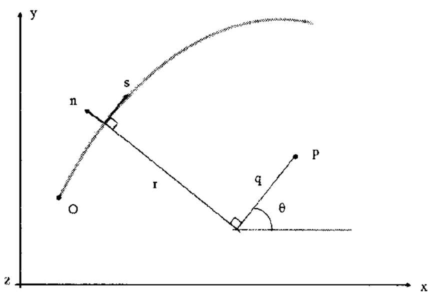

FIG. 1 Definition of coordinates in thin-walled open section

II. KINEMATICS

The theoretical developments presented in this paper require two sets of coordinate systems which are mutually interrelated. The (n, s, z) and (x, y, z) coordinate systems are related through an angle of orientation θ as defined in Fig.1.

To derive the analytical model for a thin-walled composite beam, the following assumptions are made:

1. The contour of the thin wall does not deform in its own plane.
2. The linear shear strain $\bar{\gamma}_{sz}$ of the middle surface is zero in each element.
3. The Kirchhoff-Love assumption in classical plate theory remains valid for laminated composite thin-walled beams.

The midsurface displacement components $\bar{u}, \bar{v}$ at a point A in the contour coordinate system can be expressed

$$\bar{u}(s,z) = U(z)\sin\theta(s) - V(z)\cos\theta(s) - \Phi(z)q(s) \quad (1a)$$
$$\bar{v}(s,z) = U(z)\cos\theta(s) + V(z)\sin\theta(s) + \Phi(z)r(s) \quad (1b)$$

For each element of middle surface, the shear strain become

$$\bar{\gamma}_{sz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial s} = 0 \quad (2)$$

Eq.(2) can be integrated with respect to s from the origin to an arbitrary point on the contour,

$$\bar{w}(s,z) = W(z) - U'(z)x(s) - V'(z)y(s) - \Phi'(z)\omega(s) \quad (3)$$

where differentiation with respect to the axial coordinate z is denoted by primes ($'$); W represents the average axial displacement of the beam in the z direction; x and y are the coordinates of the contour in the (x, y, z) coordinate system; and ω is the so-called sectorial coordinate or warping function given by

$$\omega(s) = \int_{s_o}^{s} r(s)ds \quad (4a)$$

The displacement components u, v, w representing the deformation of any generic point on the profile section are given by the assumption 3.

$$u(s,z,n) = \bar{u}(s,z) \quad (5a)$$
$$v(s,z,n) = \bar{v}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial s} \quad (5b)$$
$$w(s,z,n) = \bar{w}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial z} \quad (5c)$$

The von Kármán type strains, in which only the products of u, v and their derivatives are retained and all other nonlinear terms are neglected, are considered and given by

$$\epsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\Big[\Big(\frac{\partial u}{\partial z}\Big)^2 + \Big(\frac{\partial v}{\partial z}\Big)^2\Big] \quad (6a)$$
$$\gamma_{sz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial s} \quad (6b)$$

Eq.(6) can be rewritten as

$$\epsilon_z = \bar{\epsilon}_z + n\bar{\kappa}_z + n^2\bar{\chi}_z \quad (7a)$$
$$\gamma_{sz} = \bar{\gamma}_{sz} + n\bar{\kappa}_{sz} \quad (7b)$$

where

$$\bar{\epsilon}_z = \frac{\partial \bar{w}}{\partial z} + \frac{1}{2}\Big[\Big(\frac{\partial \bar{u}}{\partial z}\Big)^2 + \Big(\frac{\partial \bar{v}}{\partial z}\Big)^2\Big] \quad (8a)$$
$$\bar{\kappa}_z = -\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial^2 \bar{u}}{\partial s \partial z}\frac{\partial \bar{v}}{\partial z} \quad (8b)$$
$$\bar{\kappa}_{sz} = -2\frac{\partial^2 \bar{u}}{\partial s \partial z}; \quad \bar{\chi}_z = \Big(\frac{\partial^2 \bar{u}}{\partial s \partial z}\Big)^2 \quad (8c)$$

In Eq.(8), $\bar{\epsilon}_z$, $\bar{\kappa}_z, \bar{\kappa}_{sz}$ and $\bar{\chi}_z$ are midsurface axial strain, biaxial curvature and high order curvature of the shell, respectively. The above shell strains can be converted to beam strain components by substituting Eqs.(1), (3) and (5) into Eq.(8) as

$$\bar{\epsilon}_z = \epsilon_z^{\circ} + x\kappa_y + y\kappa_x + \omega\kappa_\omega \quad (9a)$$
$$\bar{\kappa}_z = \kappa_y\sin\theta - \kappa_x\cos\theta - \kappa_\omega q + \chi_z r \quad (9b)$$
$$\bar{\kappa}_{sz} = \kappa_{sz}; \quad \bar{\chi}_z = \chi_z \quad (9c)$$

where $\epsilon_z^{\circ}, \kappa_x, \kappa_y, \kappa_\omega, \kappa_{sz}$ and χ_z are axial strain, biaxial curvatures in the x and y direction, warping curvature with respect to the shear center, twisting and high order curvature in the beam, respectively defined as

$$\epsilon_z^{\circ} = W' + \frac{1}{2}[U'^2 + V'^2 + (r^2+q^2)\Phi'^2] - x_pV'\Phi' + y_pU'\Phi' \quad (10a)$$
$$\kappa_x = -V'' - U'\Phi'; \quad \kappa_y = -U'' + V'\Phi' \quad (10b)$$
$$\kappa_\omega = -\Phi'; \quad \kappa_{sz} = 2\Phi'; \quad \chi_z = \frac{1}{2}\Phi'^2 \quad (10c)$$

The resulting strains can be obtained from Eqs.(7) and (9) as

$$\epsilon_z = \epsilon_z^{\circ} + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega + (2rn + n^2)\chi_z \quad (11a)$$
$$\gamma_{sz} = n\kappa_{sz} \quad (11b)$$

III. VARIATIONAL FORMULATION

Strain energy of the system is calculated by

$$\mathcal{U} = \frac{1}{2}\int_v (\sigma_z\epsilon_z + \sigma_{sz}\gamma_{sz})dv \tag{12}$$

After substituting Eq.(11) into Eq.(12)

$$\mathcal{U} = \frac{1}{2}\int_v \Big\{\sigma_z\Big[\epsilon_z^\circ + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega + (2rn + n^2)\chi_z\Big] + \sigma_{sz}n\kappa_{sz}\Big\}dv \tag{13}$$

The variation of strain energy can be stated as

$$\delta\mathcal{U} = \int_0^l (N_z\delta\epsilon_z + M_y\delta\kappa_y + M_x\delta\kappa_x + M_\omega\delta\kappa_\omega + M_t\delta\kappa_{sz} + R_z\delta\chi_z)ds \tag{14}$$

where $N_z, M_x, M_y, M_\omega, M_t, R_z$ are defined by integrating over the cross-sectional area A as

$$N_z = \int_A \sigma_z dsdn \tag{15a}$$
$$M_y = \int_A \sigma_z(x + n\sin\theta)dsdn \tag{15b}$$
$$M_x = \int_A \sigma_z(y - n\cos\theta)dsdn \tag{15c}$$
$$M_\omega = \int_A \sigma_z(\omega - nq)dsdn \tag{15d}$$
$$M_t = \int_A \sigma_{sz}ndsdn \tag{15e}$$
$$R_z = \int_A \sigma_z(2rn + n^2)dsdn \tag{15f}$$

The variation of the strain energy can be obtained by substituting Eqs.(10) and (11) into Eq.(14),

$$\delta\mathcal{U} = \int_0^l \Big[N_z\delta W' - M_y\delta U'' - M_x\delta V'' - M_\omega\delta\Phi'' + 2M_t\delta\Phi' + N_z(U'\delta U' + V'\delta V') + (M_y - x_pN_z)(V'\delta\Phi' + \Phi'\delta V') - (M_x - y_pN_z)(U'\delta\Phi' + \Phi'\delta U') + (r_p^2N_z + R_z)\Phi'\delta\Phi'\Big]dz \tag{16}$$

The variation of work done by external forces

$$\delta\mathcal{V} = -\int_v (p_z\delta w + p_n\delta u + p_s\delta v)dv \tag{17}$$

where p_z, p_n, p_s are forces acting in z, n and s direction.

After substituting Eqs.(1) and (3) into Eq.(17)

$$\delta\mathcal{V} = -\int_0^l [\mathcal{P}_z\delta W + \mathcal{V}_x\delta U + \mathcal{M}_y\delta U' + \mathcal{V}_y\delta V + \mathcal{M}_x\delta V' + \mathcal{T}\delta\Phi + \mathcal{M}_\omega\delta\Phi']dz \tag{18}$$

Principle of total potential energy can be stated

$$0 = \delta\Pi = \delta\mathcal{U} + \delta\mathcal{V} \tag{19}$$

Substituting Eqs.(16) and (18) into Eq.(19) the weak form of the present theory for thin-walled composite beams are given by

$$0 = \int_0^l \Big[N_z\delta W' - M_y\delta U'' - M_x\delta V'' - M_\omega\delta\Phi'' + 2M_t\delta\Phi' + N_z(U'\delta U' + V'\delta V') + (M_y - x_pN_z)(V'\delta\Phi' + \Phi'\delta V') - (M_x - y_pN_z)(U'\delta\Phi' + \Phi'\delta U') + (r_p^2N_z + R_z)\Phi'\delta\Phi' - \mathcal{P}_z\delta W - \mathcal{V}_x\delta U - \mathcal{M}_y\delta U' - \mathcal{V}_y\delta V - \mathcal{M}_x\delta V' - \mathcal{T}\delta\Phi - \mathcal{M}_\omega\delta\Phi'\Big]dz \tag{20}$$

IV. CONSTITUTIVE EQUATIONS

The constitutive equations of a k^{th} orthotropic lamina in the laminate co-ordinate system are given by

$$\begin{Bmatrix} \sigma_z \\ \sigma_{sz} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11}^* & \bar{Q}_{16}^* \\ \bar{Q}_{16}^* & \bar{Q}_{66}^* \end{bmatrix}^k \begin{Bmatrix} \epsilon_z \\ \gamma_{sz} \end{Bmatrix} \tag{21}$$

where $\bar{Q}_{ij}^*$ are transformed reduced stiffnesses. The constitutive equations for bar forces and bar strains are obtained by using Eqs.(11), (15) and (21)

$$\begin{Bmatrix} N_z \\ M_y \\ M_x \\ M_\omega \\ M_t \\ R_z \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{sym.} & & & & & E_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_z^\circ \\ \kappa_y \\ \kappa_x \\ \kappa_\omega \\ \kappa_{sz} \\ \chi_z \end{Bmatrix} \tag{22}$$

where E_{ij} are stiffnesses of the thin-walled composite.

V. GOVERNING EQUATIONS

The nonlinear equilibrium equations of the present study can be obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of $\delta W, \delta U, \delta V$ and $\delta\Phi$

$$N_z' + \mathcal{P}_z = \tag{23a}$$
$$M_y'' + [N_z(U' + y_p\Phi')]' - [M_x\Phi']' + \mathcal{V}_x - \mathcal{M}_y' = \tag{23b}$$
$$M_x'' + [N_z(V' - x_p\Phi')]' + [M_y\Phi']' + \mathcal{V}_y - \mathcal{M}_x' = \tag{23c}$$
$$M_\omega'' + 2M_t' + [N_z(r_p^2\Phi' + y_pU' - x_pV')]' + [M_yV']' - [M_xU']' + [R_z\Phi']' + \mathcal{T} - \mathcal{M}_\omega' = \tag{23d}$$

Eqs.(23) are the general nonlinear equilibrium equations.

VI. FINITE ELEMENT FORMULATION

The present theory for thin-walled composite beams described in the previous section was implemented via a displacement based finite element method. The generalized displacements are expressed over each element as a linear combination of the one-dimensional Lagrange interpolation function Ψ_j and Hermite-cubic interpolation function ψ_j associated with node j and the nodal values

$$W = \sum_{j=1}^{n} w_j \Psi_j; \quad U = \sum_{j=1}^{n} u_j \psi_j \tag{24a}$$

$$V = \sum_{j=1}^{n} v_j \psi_j; \quad \Phi = \sum_{j=1}^{n} \phi_j \psi_j \tag{24b}$$

Substituting these expressions into the weak statement in Eq.(20), the finite element model of a typical element can be expressed as

$$[K(\{\Delta\})]\{\Delta\} = \{f\} \tag{25}$$

Solution of Eq.(25) by the Newton−Raphson iteration method results in the following linearized equations for the incremental solution at the r^{th} iteration (Ref.[19])

$$[T\{\Delta\}^{r-1}]\{\Delta\} = \{f\} - ([K]\{\Delta\}^{r-1} \tag{26}$$

where the tangent stiffness matrix is defined by

$$T_{ij}^{\alpha\beta} = \frac{\partial K_{ik}^{\alpha\gamma}}{\partial u_j^{\beta}} u_k^{\gamma} + K_{ij}^{\alpha\beta} \tag{27}$$

In Eq.(25), $\{\Delta\}$ is the unknown nodal displacements

$$\{\Delta\} = \{W \ U \ V \ \Phi\}^{\mathrm{T}} \tag{28}$$

VII. NUMERICAL EXAMPLES

For verification purpose, a cantilever composite I-beam, with warping constrained at both ends, has length l=0.762m, the cross section and the stacking sequences shown in Ref.[5] under the application of a tip torque of 0.113 Nm is considered. The angle of twist along the length using present analysis are compared with previous available results in Fig.2. The present results show better correlation with experiments than the previous results.

The next example is a cantilever composite Z-section beam with geometry and stacking sequences shown in Fig.3 is subjected to a tip shear load of 4.54 N. The results using the present analysis are compared with previously available results Refs.[4,20] in Table I. The correlation between the present analysis and previously available results is good in all cases.

A pinned-hinged composite I-beam of length $L = 8$m under an eccentric uniform load q acting at the midplane of the top flange is considered in order to investigate the effect of load parameter and fiber orientation on displacements. The geometry and stacking sequences of composite I-beam are shown in Fig.4, and the following engineering constants are used

$$E_1/E_2 = 25, G_{12}/E_2 = 0.6, \nu_{12} = 0.25 \tag{29}$$

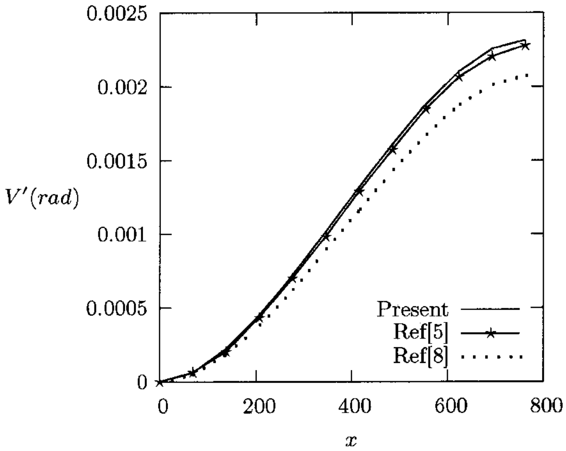

FIG. 2 Bending slope distribution along a cantilever beam subjected to a tip torque of 0.113Nm

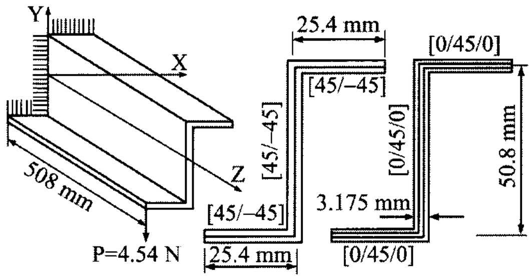

FIG. 3 A cantilever composite Z-section beam under a tip shear load

Stacking sequence of this beam consists of four layers with equal thickness as follows: $[\theta/-\theta]_2$ at bottom flange and unidirectional at web and top flange, respectively. For this stacking sequence, the coupling stiffnesses E_{15}, E_{16} and E_{35} do not vanish due to unsymmetric stacking sequence of the webs and flanges. Accordingly, the beam sustains simultaneously two kinds of couplings from material anisotropy and geometric nonlinearity.

TABLE I The tip deflection and the angle of twist of a cantilever beam under eccentric load

Lay-up	[45°/ − 45°]			[0°/45°/0°]		
	Ref.[4]	Ref.[20]	Present	Ref.[4]	Ref.[20]	Present
W	0.016	0.008	0.008	0.000	0.053	0.000
U	-3.120	-3.558	-3.521	-2.390	-2.638	-2.596
U'	-9.210	-10.510	-10.396	-7.070	-7.791	-7.664
V	2.090	2.372	2.356	1.610	1.759	1.738
V'	6.190	7.003	6.958	4.750	5.194	5.133
Φ	27.050	34.930	35.255	29.250	30.890	30.773

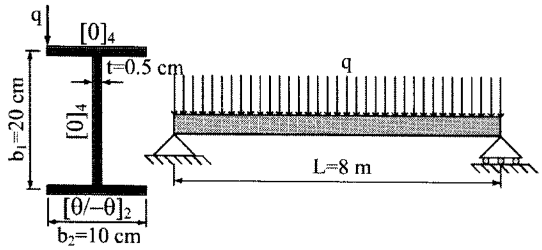

FIG. 4 A pinned-hinged composite I-beam under an eccentric uniform load

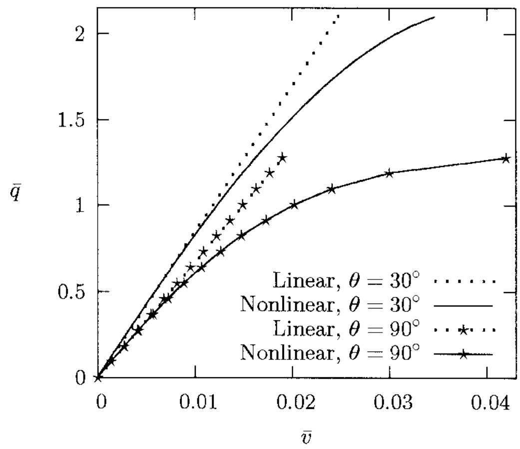

FIG. 5 Load versus vertical displacement at mid-span.

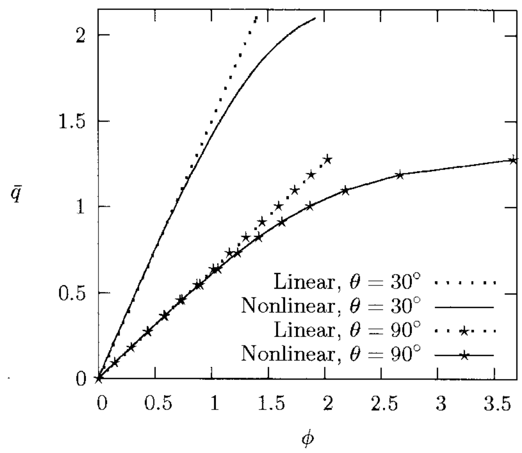

FIG. 6 Load versus the angle of twist at mid-span.

As a first example, stacking sequence at two specific fiber angle $\theta = 30°, 90°$ is considered to investigate the effects of load parameter on displacements in the high nonlinear region. It should be noted that for $\theta = 90°$, all the coupling stiffness vanish, that is, only geometrical nonlinear effect exists. The load with increment of $\Delta\bar{q} = 0.05$ is increased until the first critical point is obtained. Figs.5 and 6 show the load versus vertical displacement, the angle of twist of two stacking sequences. The highest load for fiber angle $\theta = 90°$ is smaller than that of $\theta = 30°$. It is evident that the linear theory is adequate in a relatively large region up to the point where applied load reaches value of $\bar{q} = 0.5, 1$ for fiber angle $\theta = 90°$ and $\theta = 30°$, respectively. It is clear that the nonlinear analysis predicts a softer response in vertical direction and flexibility in rotation. This is due to the fact that the geometrical nonlinear effect causes flexural-torsional coupling which results in an increase in the flexural stiffness and a decrease in the torsional stiffness of the beam. For $\theta = 30°$, at the highest load presented in Figs.5,6 the nonlinear vertical displacement is about 140% of the linear value and the nonlinear angle of twist is about 138% of the linear value. It is observed that the effect of the geometric nonlinearity is apparent with increasing load intensity. This implies that discarding this effect leads to an overprediction of displacements.

To investigate the geometrical nonlinear effect further, a fix load is considered while the fiber angle is rotated in the webs. Based on previous numerical example, an applied load $\bar{q} = 1.0$ is chosen to show effect of fiber angle on the flexural-torsional response. Variation of the vertical and torsional displacement with respect to fiber angle change are shown in Figs.7 and 8. It appears that the nonlinear vertical displacement are not as sensitive as the nonlinear torsional displacement when fiber angle changes. Especially, for fiber angles less than $\theta = 30°$, the vertical displacement of linear and nonlinear analysis coincides. As the fiber orientation is rotated off-axis, geometrical nonlinear effect is prominent, that is, the discrepancy between the nonlinear and linear analysis becomes significant. The difference between displacements of two analyses is minimum at $\theta = 0°$ and reaches maximum value at $\theta = 90°$. This phenomenon can be explained that the axial, flexural and torsional rigidities decrease significantly, consequently, the relative geometrical nonlinear effect increases when comparing to that at $\theta = 0°$.

VIII. CONCLUDING REMARKS

The effect of geometric nonlinearity on flexural-torsional response of thin-walled open-section composite beams under general external loading is presented. The nonlinear algebraic equations of present theory are linearized and solved by means of an incremental Newton–Raphson method. Based on the analytical model, a displacement-based one-dimensional finite element model that accounts for the geometric nonlinearity in the von Kármán sense is developed to formulate the problem. The inclusion of geometric nonlinear effect is required for the cases of composite beams subjected to

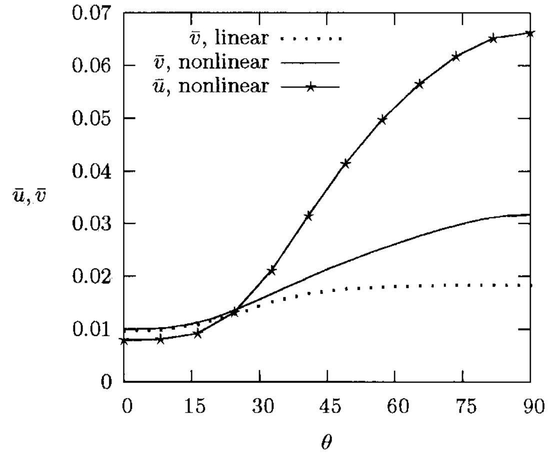

FIG. 7 Variation of the horizontal and vertical displacements at mid-span with respect to fiber angle change.

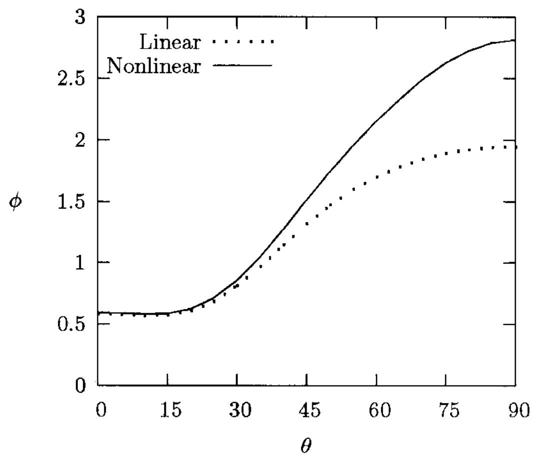

FIG. 8 Variation of the angle of twist at mid-span with respect to fiber angle change.

high loading. The model presented is found to be appropriate and efficient in analyzing geometrically nonlinear behavior of thin-walled open-section composite beams.

Acknowledgments

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