Derivation of Closed Form Channel Capacity Using Confluent Hypergeometric Function for Wireless MIMO

S. M. Humayun Kabir, Van-Su Pham, Linh Mai and Giwan Yoon, Member KIMICS

Information and Communications University (ICU)

119 Munji-ro, Yuseong-gu, Daejeon 305-732, R.O. Korea

kabir@icu.ac.kr

Abstract - Multiple-input multiple-output (MIMO) is an efficient technology to increase data rate in wireless networks due to bandwidth and power limitations. Data transmission rate between transmitter and receiver is determined by channel capacity. MIMO has anadvantage of reliable communication over wireless channel because of utilizing the channel capacity properly. In this letter, we drive a new formula, closed form capacity formula, using confluent hypergeometric function.

Index Terms— MIMO systems, channel capacity.

I. Introduction

Demand for capacity in wireless communications have been rapidly increasing worldwide, and as a solution to fulfill such a demand, multiple transmitting and receiving antennas namely, multiple-input multiple-output (MIMO) systems have been introduced. The MIMO systems has replaced the single-input single-output (SISO) systems mainly because the multiple antennas at the transmitter and the receiver canincrease data rate and improve quality of service through diversity. Channel capacity is used to determine the consistent information transmission. The standard formula for the Shannon channel capacity is expressed in bps/Hz [1]

$$C = \log_2\left(1 + \rho \left|H\right|^2\right)$$

where, $|H|^2$, denotes the normalized channel power transfer characteristic. To achieve higher data rate research has been done based on this formula. The unequal number of antennas at both the transmitter and receiver increase the capacity logarithmically whereas the equal number of antennas increases the capacity linearly for any fixed signal-to-noise ratio [1] [2].Multi-element array can effectively comprehend a MIMO wireless channel. Considering the performance of the two situations, the capacity of single user multi-element arrays systems can be explored as follows.

Firstly, the channel is known to the transmitter, thus the power allocation is thefinest; secondly, the channel is totally unknown to the transmitter. So, equal power is allocated for each transmitting antenna [3].

In this letter using the confluent hypergeometric function we have introduced an analytical formula for the channel capacity of MIMO wireless systems. In section II we have describe the MIMO system model. In section III proposedchannel capacity formula has derived, followed by simulation results in section 4. Finally, section 5 concludes the paper.

II. SYSTEM MODEL

We consider a single point-to-point MIMO system where the number of transmit antennas are expressed as n_t and the number of receive antennas as n_r. We also assume that $n = max(n_t, n_r)$ and $m = min(n_t, n_r)$. The transmitted signals in each symbol period are represented by an $n_t \times 1$ vectors s, whose i^{th} entry, s_i, will be transmitted from antenna i.We consider the total transmitted power over a symbol period to P, not considering the number of transmit antennas n_t. It is assumed that the covariance matrix of s, $R_{ss} = \varepsilon\{ss^H\}$, satisfies the criterion of

$$\mathrm{Tr}(\mathbf{R}_{ss}) \le P \tag{2}$$

where $Tr(.)$ denotes the trace of a matrix.

Under the assumption that the channel is frequency-flat fading, the input-output relation over a symbol period is given by

$$\mathbf{r} = \mathbf{Hs} + \mathbf{w} \tag{3}$$

where,

- $\mathbf{r}$ is the $n_r \times 1$ receive vector
- $\mathbf{H}$ is the $n_r \times n_t$ channel matrix whose entries are considered the corresponding path gain between the transmit and receive antennas.
- $\mathbf{w}$ represents the additive noise at n_r receive antennas.

The transmitter completely has no knowledge of the channel but the receiver has perfect knowledge of the channel and the transmit power from each antenna is P/n_t. The overall channel capacity can be determined by using Shannon capacity formula [4]

$$C = W\sum_{i=1}^{r}\log_2\left(1+\frac{P_{ri}}{\sigma^2}\right) \tag{4}$$

where σ^2 is the noise power, W is the bandwidth of each sub-channel and P_{ri} is the received signal power in the ith sub-channel. It is given by

$$P_{ri} = \frac{\lambda_i P}{n_t} \tag{5}$$

where $\sqrt{\lambda_i}$ is the singular value of the channel matrix $\mathbf{H}$. Thus the channel capacity can be written as [4]

$$C = W\log_2 \det(\mathbf{I}_m + \frac{P}{n_t\sigma^2}\mathbf{Q}) \tag{6}$$

when the transmitted signal vector s is circularly symmetric complex Gaussian with zero-mean and covariance $(P/n_t)I_{n_t}$ and then the capacity is given by [2]

$$C = \varepsilon\{\log_2 \det\left(\mathbf{I}_{r_t} + \frac{P}{n_t\sigma^2}\mathbf{Q}\right)\} \tag{7}$$

where the random matrix $\mathbf{Q}$ is theWishart matrix defined as

$$\mathbf{Q} = \begin{cases} HH^h, & n_r < n_t \\ H^hH, & n_r \geq n_t \end{cases} \tag{8}$$

Again in [2], the capacity of the channel with n_t transmitters and n_r receivers under power constraint P is expressed as

$$C = \int_0^\infty \log\left(1+\frac{P\lambda}{n_t}\right)\sum_{k=0}^{m-1}\frac{k!}{(k+n-m)!} \times [L_k^{n-m}(\lambda)]^2\lambda^{n-m}e^{-\lambda}d\lambda \tag{9}$$

where L_i^j is the associated Laguerre polynomials defined as [5]

$$L_a^b(x) = \frac{e^x x^{-b}}{a!}\frac{d^a(e^{-x}x^{a+b})}{dx^a} \tag{10}$$

Also in [5], the connection between Laguerre polynomials and confluent hypergeometric function of the first kind is given by

$$L_a^b(x) = \frac{\Gamma(a+b+1)}{a!\,\Gamma(b+1)}\ 1F1(-a;b+1;x) \tag{11}$$

where the hypergeometric function has a hypergeometric series given by

$$1F1(a;b;z) = 1+\frac{a}{b}z+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2}+\cdots = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{(z)^k}{k!} \tag{12}$$

where $(a)_k$ and $(b)_k$ are Pochhammer symbols. If a and b are integers, $a < 0$, and either $b > 0$ or $b < a$, then the series yields a polynomial with a finite number of terms, *i.e.* it turns to a closed form.

III. DERIVED FORMULA FOR CAPACITY

We drive the formula for the channel capacity. The Laguerre polynomial is defined as [6]

$$[L_a^b(x)]^2 = \frac{\Gamma(a+b+1)}{2^{2a}a!}\sum_{k=0}^{a}\binom{2a-2k}{a-k}\frac{(2k)!}{k!} \times\frac{1}{\Gamma(b+k+1)}L_{2k}^{2b}(2x) \tag{13}$$

where $\Gamma(.)$ is the gamma function.

According to (13), the Laguerre polynomial of (9) can be expressed as

$$[L_k^{n-m}(\lambda)]^2 = \frac{\Gamma(k+n-m+1)}{2^{2k}k!}\sum_{l=0}^{k}\binom{2k-2l}{k-l}\frac{(2l)!}{l!} \times\frac{1}{\Gamma(n-m+l+1)}L_{2l}^{2n-2m}(2\lambda) \tag{14}$$

Expressing $L_{2l}^{2n-2m}(2\lambda)$ in terms of (11), we get

$$L_{2l}^{2n-2m}(2\lambda) = \frac{\Gamma(2n-2m+2l+1)}{(2l)!\,\Gamma(2n-2m+1)} \times 1F1(-2l; 2n-2m+1; 2\lambda) \qquad (15)$$

According to (12), the hypergeometric function $1F1(-2l; 2n-2m+1;$ will be

$$1F1(-2l; 2n-2m+1; 2\lambda) = \sum_{f=0}^{\infty} \frac{(-2l)_f}{(2m-2l+1)_f} \frac{(2\lambda)^f}{f!} \qquad (16)$$

Replacing (16) into (15), we get

$$L_{2l}^{2n-2m}(2\lambda) = \frac{\Gamma(2n-2m+2l+1)}{(2l)!\,\Gamma(2n-2m+1)} \times \sum_{f=0}^{\infty} \frac{(-2l)_f}{(2m-2l+1)_f} \frac{(2\lambda)^f}{f!} \qquad (17)$$

Substituting (17) into (14), we get

$$[L_k^{n-m}(\lambda)]^2 = \frac{\Gamma(k+n-m+1)}{2^{2k}k!} \sum_{l=0}^{k} \sum_{f=0}^{\infty} \binom{2k-2l}{k-l} \times \frac{1}{\Gamma(n-m+l+1)} \frac{\Gamma(2n-2m+2l+1)}{l!\,\Gamma(2n-2m+1)} \times \frac{(-2l)_f}{(2m-2l+1)_f} \frac{(2\lambda)^f}{f!} \qquad (18)$$

Considering the relation $\Gamma(n) = (n-1)!$, and substituting (18) into (9) we get the associated channel capacity that can be expressed as

$$C = \log_2(e) \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{f=0}^{\infty} \binom{2k-2l}{k-l} \times \frac{1}{(n-m+l)!} \frac{(2n-2m+2l)!}{(2n-2m)!\,l!} \frac{(-2l)_f}{(2m-2l+1)_f} \times \frac{2^f}{2^{2k} f!} \int_0^{\infty} \ln\left(1+\frac{P\lambda}{n_t}\right) \lambda^{n-m+f} e^{-\lambda} d\lambda \qquad (19)$$

For computing the integral of the above equation, we use the result of appendix B of [5]

$$I_m(\alpha) = \int_0^{\infty} \ln(1+y)\, x^{m-1} e^{-\alpha y}\, dy, \qquad \alpha > 0,1,2,\ldots$$
$$= (m-1)!\, e^{\alpha} \sum_{j=1}^{m} \frac{\Gamma(-m+j, \alpha)}{\alpha^j} \qquad (20)$$

where the complementary incomplete gamma function is defined in [6] as

$$\Gamma(\alpha, z) = \int_z^{\infty} e^{-y} y^{\alpha-1} dy \qquad (21)$$

Computing the integral of (19), according to (20)

$$I = (n-m+f)! \left(\frac{n_t}{P}\right)^{n-m+f+1} e^{\left(\frac{n_t}{P}\right)} \times \sum_{i=1}^{n-m+f+1} \frac{\Gamma(-n+m-f-1+i, n_t/P)}{(n_t/P)^{-i}} \qquad (22)$$

In [7], the relation between the exponential integral of order n and the complementary incomplete gamma function is defined as

$$E_p(x) = x^{p-1}\Gamma(1-p, x) \qquad (23)$$

By employing (23) to (22), we get

$$I = (n-m+f)!\, e^{(n_t/P)} \sum_{i=0}^{n-m+f} E_{i+1}(n_t/P) \qquad (24)$$

Inserting (24) into (19), and also applying the condition of (12), we get the following equation:

$$C = e^{(n_t/P)} \log_2(e) \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{f=0}^{2l} \binom{2k-2l}{k-l} \times \frac{(n-m+f)!}{(n-m+l)!} \frac{(2n-2m+2l)!}{(2n-2m)!\,l!} \frac{(-2l)_f}{(2m-2l+1)_f} \times \frac{1}{2^{2k}} \frac{2^f}{f!} \sum_{i=0}^{n-m+f} E_{i+1}(n_t/P) \qquad (25)$$

Figure 1 shows simulation result of the derived capacity. Simulation is done for the different number of antennas in both transmitter and receiver side. Simulation result shows that the capacity increased logarithmically because of unequal number of antennas are deployed in both the transmitter and receiver.

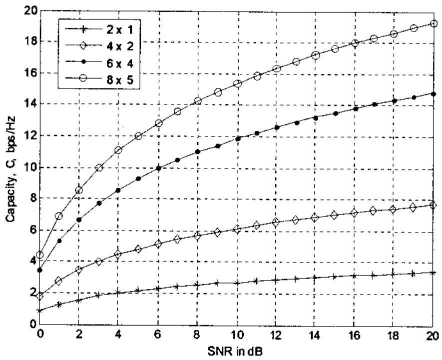

Figure 1. Analytical channel capacity versus average signal to noise ratio. The legend denotes the number of transmit and receive antennas.

IV. CONCLUSION

In this paper, a closed form channel capacity formula has been derived using confluent hypergeometric function. Simulation was performed to get analytical channel capacity versus average signal to noise ratio. The derived capacity formula allows us to calculate the channel capacity in a closed form. Consequently, the derived formula can be applied for calculating the channel capacity of practical MIMO system where large numbers of antennas are used to transmit and receive data through wireless media.

V. Acknowledgement

This work was supported by the ERC program of MOST/KOSEF (Intelligent Radio Engineering Center).

VI. References

[1] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless personal Communications*, vol. 6, pp. 311-335, 1998.

[2] I.E. Teletar, "Capacity of multi antenna Gaussian channel", *European Trans. Telecomm. (ETI)*, vol.10, no.6 pp.586-595, Nov./Dec. 1999.

[3] C.N. Chuah, D.N.C. Tse, and J.M. Kahn, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Trans. Inf. Theory*, vol.48, no.3, pp.637–650, March 2002.

[4] B. Vucetic and J. Yuan, *Space-Time coding*, John Wiley & Sons, 2003

[5] A. Erdélyi, "Transformation of a certain series of products of confluent hypergeometric functions: Applications to Laguerre and Charlier polynomials," Compos. Math. 7 (1939) 340-352.

[6] I.S Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., Academic, San Diego, Ca, 2000.

[7] H. Shin and J. H. Lee. "On the capacity of MIMO wireless channels," *IEICE Trans. Commun.*, Vol. E87-B, no.3, pp. 671-677, Mar.2004.