BMAP/M/N/0 Queueing System in Random Environments

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Abstract. The *BMAP/M/N/0* queueing system operating in Markovian random environment is investigated. The stationary distribution of the system is calculated. Loss probability and other performance measures are calculated. Numerical experiments which show the necessity of taking into account the influence of random environment and correlation in input flow are presented.

Key words: BMAP/M/N/0 queueing model, stationary state distribution, loss probability.

1. Introduction

At the beginning of 20th century Danish engineer A. K. Erlang laid the foundation of the queueing theory. He investigated the queueing system *M/M/N/0* which was a good mathematical model of that time telephone networks. Erlang's formula for a loss probability in the *M/M/N/0* queue serves for needs of the practical engineering until now.

However, the flows in the modern telecommunication networks have lost the nice properties of their predecessors in the old classic networks. In opposite to the stationary Poisson input (stationary ordinary input with no aftereffect), the modern real life flows are non-stationary, group and correlated. The *BMAP* (Batch Markovian Arrival Process) was introduced by D. Lucantoni [1] as one of the appropriate models of such inputs. The Erlang loss model for the case of the *BMAP* input was investigated in [2]. The numerical experiments presented in [2] show an essential difference of loss probability in *M/M/N/0* and *BMAP/M/N/0*queueing systems with equal intensity of input flows. That confirms the necessity to take into account the correlated bursty traffic in modern telecommunication networks. The extension of the *BMAP/M/N/0* model to the case of *PH* (Phase-type) service time distribution was considered in [3].

In the present paper we extend *BMAP/M/N/0* model assuming that the system has *R* different modes of operations, and the modes are switched by an external random process, so called random environment.

The considered queueing model has a wide range of potential applications because in practical systems the input and service processes are not absolutely stable, they are influenced by external factors, e.g., the different level of the noise in the transmission channel, hardware degradation and recovering, change of the distance by a mobile user from the base station, etc.

2. Mathematical Model

We consider an *N*-server queueing system. The behavior of the system depends on the state of the stochastic process (random environment) r_t, $t \geq 0$, which is assumed to be an irreducible continuous time Markov chain with the state space $\{1,\ldots,R\}$, $R \geq 2$, and the infinitesimal generator Q.

The input flow into the system is the following modification of the well-known (see, e.g., [1]) BMAP. In this input flow, the batch arrivals are directed by the process ν_t, $t \ge 0$ (the directing process) with the state space $\{0,1,\ldots,W\}$. Under the fixed state $r$of the random environment, this process behaves as an irreducible continuous time Markov chain. Transitions of the chain ν_t, $t \ge 0$, which are not accompanied by arrival, are described by the matrix $D_0^{(r)}$, and transitions, which are accompanied by arrival of k-size batch, are described by the matrix $D_k^{(r)}$, $k \ge 1$, $r = \overline{1,R}$. The matrix $D^{(r)}(1)$ is an irreducible generator for all $r = \overline{1,R}$. Under the fixed state r of the random environment, the average intensity $\lambda^{(r)}$ (fundamental rate) of the *BMAP* is defined as $\lambda^{(r)} = \boldsymbol{\theta}^{(r)}(D^{(r)}(z))'|_{z=1}\,\mathbf{e}$, and the intensity $\lambda^{(r)}$ of batch arrivals is defined as $\lambda^{(r)} = \boldsymbol{\theta}^{(r)}(-D_0^{(r)})\mathbf{e}$. Here $\boldsymbol{\theta}^{(r)}$ is the solution to the equations $\boldsymbol{\theta}^{(r)}D^{(r)}(1) = \mathbf{0}$, $\boldsymbol{\theta}^{(r)}\mathbf{e} = 1$, $\mathbf{e}$ is a column vector of appropriate size consisting of 1's. The variation coefficient $c_{\mathrm{var}}^{(r)}$ of intervals between batch arrivals is given by $(c_{\mathrm{var}}^{(r)})^2 = \lambda^{(r)}\boldsymbol{\theta}^{(r)}(-D_0^{(r)})^{-1}\mathbf{e} - 1$, while the correlation coefficient $c_{cor}^{(r)}$ of intervals between successive batch arrivals is calculated as $c_{cor}^{(r)} = (\lambda^{(r)}\boldsymbol{\theta}^{(r)}(-D_0^{(r)})^{-1}(D^{(r)}(1) - D_0^{(r)})(-D_0^{(r)})^{-1}\mathbf{e} - 1)/(c_{cor}^{(r)})^2$. The state of the process ν_t, $t \ge 0$, is not changed at the epochs of the process r_t, $t \ge 0$, transitions.

The system under consideration has no waiting space. So, if the system has all servers being busy at a batch arrival epoch, the batch leaves the system forever and considered to be lost. If there are free servers at arrival epoch, however the number of these servers is less than the number of customers in the group so called partial admission discipline is used. It means that only a part of the group corresponding to a number of free servers is allowed to enter the service while the rest of the group is lost. It is assumed that all servers are identical and operate independently of each other. Service time of a customer by a server has an exponential distribution of intensity $\mu^{(r)}$ under the state r of the random environment. Our aim is to calculate the stationary state distribution and main performance measures of the described queueing model.

3. Stationary State Distribution

It is easy to see that operation of the considered queueing model is described in terms of the irreducible continuous-time Markov chain $\xi_t = \{i_t, r_t, \nu_t\}$, $t \ge 0$, where i_t is the number of customers in the system (the number of busy servers), r_t is the state of random environment, $r_t = \overline{1,R}$, and ν_t is the state of the *BMAP* directing process at the epoch t, $t \ge 0$. Enumerate the states of the chain ξ_t, $t \ge 0$ in the lexicographic order and form the row vectors $\mathbf{p}_i$, $i = \overline{0,N}$ of probabilities corresponding to the state i of the first component of the process ξ_t, $t \ge 0$. Denote also $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_N)$.

The vector $\mathbf{p}$ satisfies the system of linear algebraic equations of the form:

$$\mathbf{p}A = 0, \quad \mathbf{p}\mathbf{e} = 1, \tag{1}$$

where A is an infinitesimal generator of the Markov chain ξ_t, $t \ge 0$.

Let I be an identity matrix of size listed as the low index, $I_0 = 1$; $\otimes$ and $\oplus$ be the symbols of Kronecker's product and sum of matrices; $D_i = diag\{D_i^{(r)},\ r = \overline{1,R}\},\ i = \overline{0, N-1}$; $D_{N,l} = diag\{\sum_{k=l}^{\infty} D_k^{(r)},\ r = \overline{1,R}\},\ l = \overline{0,N}$; $\mu = diag\ \{\mu^{(r)},\ r = \overline{1,R}\}$.

Lemma 1. Infinitesimal generator A of the Markov chain has the following block structure ξ_t, $t \ge 0$,:

$$A = (A_{n,n'})_{n,n'=\overline{0,N}} = \tag{2}$$

$$= \begin{pmatrix} D_0 + Q \otimes I_{\overline{W}} & D_1 & D_2 & \cdots & D_{N,N} \\ \mu I_{\overline{W}} & D_0 + Q \otimes I_{\overline{W}} - \mu I_{\overline{W}} & D_1 & \cdots & D_{N,N-1} \\ 0 & 2\mu I_{\overline{W}} & D_0 + Q \otimes I_{\overline{W}} - 2\mu I_{\overline{W}} & \cdots & D_{N,N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_{N,0} + Q \otimes I_{\overline{W}} - N\mu I_{\overline{W}} \end{pmatrix}.$$

To solve system (1) we use the effective stable procedure basing on the special structure of the matrix A (it is upper block hessenbergian) and probabilistic meaning of unknown vector $\mathbf{P}$. Such a procedure was done in [3]. It is briefly described in the following statement.

Proposition 1. The stationary probability vectors $\mathbf{p}_i$, $i = \overline{0,N}$, are calculated as follows:

$\mathbf{p}_l = \mathbf{p}_0 F_l,\ \ l = \overline{1,N}$,

where the matrices F_l are calculated recurrently

$$F_l = (\overline{A}_{0,l} + \sum_{i=1}^{l-1} F_i \overline{A}_{i,l})(-\overline{A}_{l,l})^{-1},\ \ l = \overline{1,N-1},\ F_N = (A_{0,N} + \sum_{i=1}^{N-1} F_i A_{i,N})(-A_{N,N})^{-1},$$

the matrices $\overline{A}_{i,N}$ are calculated from the backward recursions

$$\overline{A}_{i,N} = A_{i,N},\ \ i = \overline{0,N},\ \overline{A}_{i,l} = A_{i,l} + \overline{A}_{i,l+1} G_l,\ \ i = \overline{0,l},\ \ l = \overline{N-1,0},$$

the matrices G_i, $i = \overline{0, N-1}$ are calculated from the backward recursion

$$G_i = (-A_{i+1,i+1} - \sum_{l=1}^{N-i-1} A_{i+1,i+1+l} G_{i+l} G_{i+l-1} \ldots G_{i+1})^{-1} A_{i+1,i},\ \ i = \overline{N-1,0},$$

the vector $\mathbf{p}_0$ is calculated as the unique solution to the following system of linear algebraic equations:

$$\mathbf{p}_0 \overline{A}_{0,0} = 0,\ \ \mathbf{p}_0(\sum_{l=1}^{N} F_l \mathbf{e} + \mathbf{e}) = 1.$$

4. Performance Measures

Having the vector $\mathbf{P}$ be calculated, we are able to calculate the main performance measure of the considered model. It is the probability P_{loss} that an arbitrary customer is lost in the system (loss probability).

Theorem 1. Loss probability P_{loss} is calculated as follows:

$$P_{loss} = 1 - \frac{1}{\lambda}\sum_{i=1}^{N} \mathbf{p}_i Q_{i,i-1}\mathbf{e}.$$

In trivial way we can also to calculate a number of other stationary performance measures of the considered model.

– The probability to see I busy servers $\mathbf{p}_i = \mathbf{p}_i\mathbf{e}, \quad i = \overline{0,N}.$

– The mean number of busy servers $N_{busy} = \sum_{i=1}^{N} i\mathbf{p}_i\mathbf{e}.$

– The joint probability to see i busy servers, the random environment in the state r and the process ν_t in the state ν

$$p(i,r,\nu) = \mathbf{p}_i \begin{pmatrix} \mathbf{0}_{r-1} \\ 0 \\ \mathbf{0}_{R-r} \end{pmatrix} \otimes \mathbf{e}_{\overline{W}}, \quad i = \overline{0,N}, \quad r = \overline{1,R}, \quad \nu = \overline{0,W},$$

where $\mathbf{e}_n$ and $\mathbf{0}_n$ are n-dimensional column vectors consisting of units and zeros respectively.

– The joint probability to see i busy servers and the random environment in the state r

$$p_i(r) = \sum_{\nu=0}^{\overline{W}} p(i,r,\nu), \quad i = \overline{0,N}, \quad r = \overline{0,R}.$$

5. Numerical Examples

Present the results of two experiments. The goal of the first experiment is confirmation of some intuitively clear reasoning relating the possibility of approximation of the system operating in random environment (system in RE).

The first type approximation is described as follows. Consider *R BMAP/M/N/0* systems. Parameters of the r-th system are defined by the parameters of the r-th operation mode. To approximate some performance characteristic of the system in RE we calculate the same characteristic for each of *R* systems without RE and then average them according to the stationary distribution of the RE.

The second type approximation is described as follows. The approximated characteristic is calculated as the corresponding characteristic of an averaged *BMAP/M/N/0* system. The parameters of this system are obtained by means of averaging the corresponding parameters of the initial system in RE according to the stationary distribution of the RE. Figure 1 illustrates the dependence of the value P_{loss} for the system in RE and the samevalue calculated by the first type approximation ("mixed system") and by the second type approximation ("mixed parameters") on the RE rate. We define the random environment with different rates by their generators having the form $Q^{(k)} = Q^{(0)} \cdot 10^k$, where the generator $Q^{(0)}$ describes the RE whose rates are comparable with the rates of input and service processes. We vary the parameter k from -5 to 5 what corresponds to the variation of the RE rate from "very slow" (comparing the rates of input flow, service and retrial processes) to "very fast".

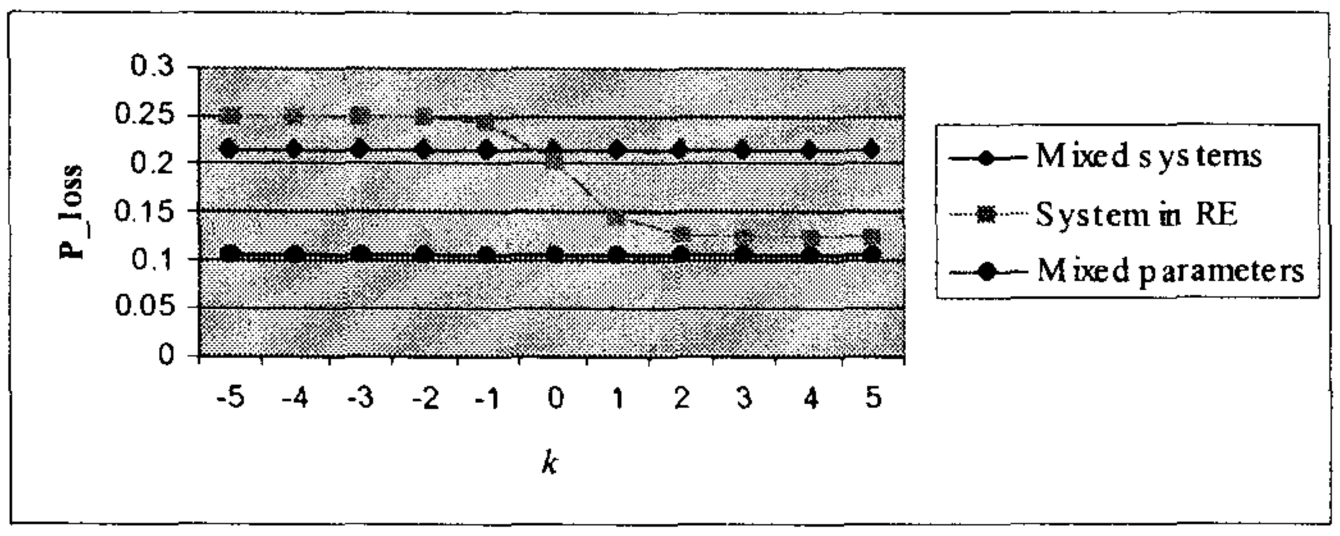

Fig.1. Dependence of the loss probability on the RE rate

Figure 1 shows that the first type approximation is good in case of "slow RE" and the second one can be applied in case of "fast RE". And there is an interval for RE rate (approximately the interval (-1,2)) where we cannot use the estimates for P_{loss} calculated by the considered approximating models. In the second experiment we compare the main performance measures of the original system in RE and more simple exponential queueing systems, which can be considered as "engineer" approximations of the original system. The first type approximate model is the system *M/M/N/0* in the RE. It differs from the original system by assumption that input flows in its modes are stationary Poisson ones whose intensities are equal to fundamental rate of corresponding *BMAP*s in the original system. The second type approximate model is the Erlang system *M/M/N/0* whose parameters are obtained by means of averaging the corresponding parameters of just described system *M/M/N/0* in the RE according to the stationary distribution of the RE.

Figure 2 shows the dependence of the loss probability (mean number of busy servers) on the system load in the original system (curve 1), in the first type approximate model (curve 2) and in the second type approximate model (curve 3).

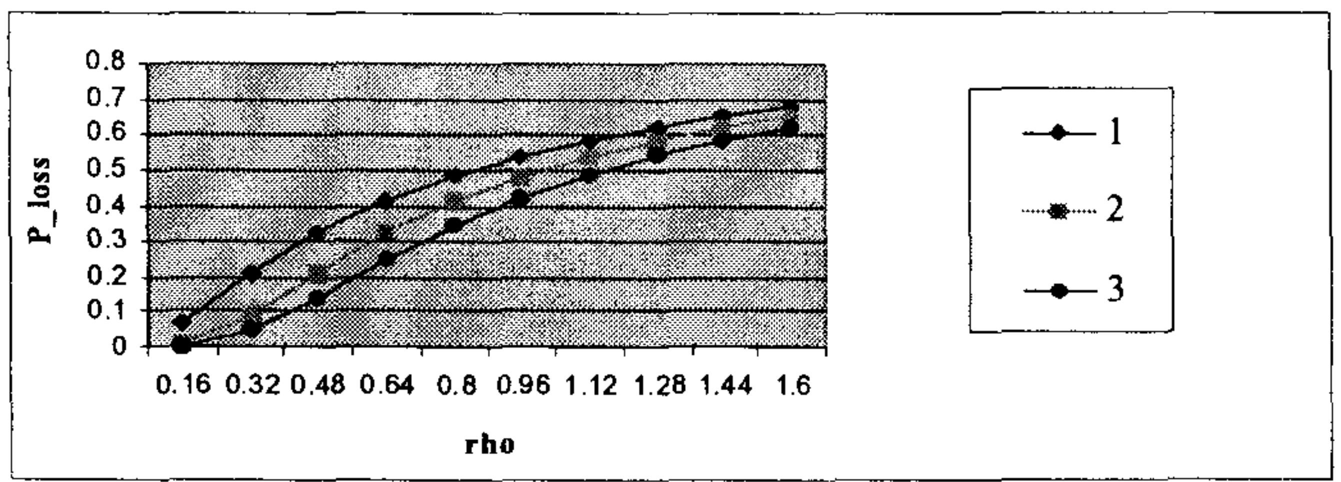

Fig.2. Dependence of the loss probability in the system in RE and in the exponential system on the system load

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