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단조집합함수에 의해 정의된 구간치 쇼케이적분에 대한 르베그형태 정리에 관한 연구

On Lebesgue-type theorems for interval-valued Choquet integrals with respect to a monotone set function.

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Abstract

In this paper, we consider Lebesgue-type theorems in non-additive measure theory and then investigate interval-valued Choquet integrals and interval-valued fuzzy integral with respect to a additive monotone set function. Furthermore, we discuss the equivalence among the Lebesgue's theorems, the monotone convergence theorems of interval-valued fuzzy integrals with respect to a monotone set function and find some sufficient condition that the monotone convergence theorem of interval-valued Choquet integrals with respect to a monotone set function holds.

Key words : monotone set functions, interval-valued functions, Choquet integrals, fuzzy integrals, Lebesgue's theorems, monotone convergence theorems.

1. Introduction

We consider both interval-valued Choquet integral[1,2,3,6] and interval-valued fuzzy integral [5] with respect to a monotone set function. Set-valued Choquet integrals was introduced by Jang and Kwon([1]) and restudied by Zhang, Guo and Lia([6]) and that the theory about set-valued integrals has drawn much attention due to numerous applications in mathematics, economics, theory of control and many other fields. Set-valued fuzzy integral was first defined by D. Zhang and Z. Wang[4]. we note that Lebesgue's theorems asserts that almost everywhere convergence implies convergence in measure on a measurable set of finite measure.

In this paper, we consider Lebesgue-type theorems for interval-valued functions in non-additive measure theory and then investigate interval-valued Choquet integrals and interval-valued fuzzy integral with respect to a additive monotone set function. Furthermore, we discuss the equivalence among the Lebesgue's theorems, the monotone convergence theorems of interval-valued fuzzy integral with respect to a monotone set function and find some sufficient condition that the monotone convergence theorem of interval-valued Choquet integrals with respect to a monotone set function holds.

2. Preliminaries

Let X be a set, (X,Ω) a measurable space and $\mathbb{F}$ the class of all finite non-negative measurable functions on X. A set function $\mu : \Omega \to R^+ = [0, +\infty)$ is said to be monotone if $\mu(A) \le \mu(B)$, whenever $A, B \in \Omega$ and $A \subset B$; null-additive if $\mu(A \cup F) = \mu(A)$ for any $A \in \Omega$ whenever $F \in \Omega$ and $\mu(F) = 0$; continuous from below if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subset \Omega$ and $A_n \nearrow A$; continuous from above if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subset \Omega$, $A_n \searrow A$ and $\mu(A_1) < \infty$; strongly order continuous if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $\{A_n\} \subset \Omega$, $A_n \searrow B$ and $\mu(B) = 0$; pseudo-order continuous if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A \in \Omega$, $\{A_n\} \subset \Omega$, $A_n \searrow B$ and $\mu(A - B) = \mu(A)$. We note that if μ is both continuous from below and continuous from above, then it is continuous. In this paper, we always assume that μ is a monotone set function with $\mu(\varnothing) = 0$.

Definition 2.1 Let $f \in \mathbb{F}$ and $\{f_n\} \subset \mathbb{F}$. $\{f_n\}$is said to converge to f almost everywhere (resp. pseudo-almost everywhere) on A if there is a subset $E \subset A$ such that $\mu(E) = 0$(resp. $\mu(A - E) = \mu(A)$) and f_n converges to f on $A - E$.

Definition 2.2 Let $f \in \mathbb{F}$ and $\{f_n\} \subset \mathbb{F}$. $\{f_n\}$is said to converge to f in measure μ(resp. pseudo-in measure μ) on A if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \ge \epsilon\} \cap A) = 0$$

$$(\text{resp.} \lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| < \epsilon\} \cap A) = \mu(A)).$$

Definition 2.3 ([3]) (1) The Choquet integral of a measurable function f with respect to a monotone set function μ on $A \in \Omega$ is defined by

$$(C)\int_A f d\mu = \int_0^\infty \mu(\{x | f(x) > r\} \cap A) dr$$

where the integrand on the right-hand side is an ordinary one.

(2) A measurable function f is called c-integrable if the Choquet integral of f can be defined and its value is finite.

Definition 2.4 ([7]) The fuzzy integral of a measurable function f with respect to a monotone set function μ on $A \in \Omega$ is defined by

$$(F)\int_A f d\mu$$
$$= \sup_{\alpha \in [0,\infty)} [\alpha \wedge (A \cap \{x | f(x) > \alpha\})].$$

3. Convergence of integral sequence

We denote $I(R^+)$ by

$I(R^+) = \{\bar{a} = [a^-, a^+] | a^- \leqq a^+, a^-, a^+ \in R^+\}$.

For any $a \in R^+$, we define $a = [a, a]$. Obviously, $a \in I(R^+)$.

Definition 3.1 If $\bar{a}, \bar{b} \in I(R^+)$, then we define

(1) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,

(2) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,

(3) $\bar{a} \le \bar{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$,

(4) $\bar{a} < \bar{b}$ if and only if $\bar{a} \le \bar{b}$ and $\bar{a} \ne \bar{b}$,

(5) $\bar{a} \subset \bar{b}$ if and only if $b^- \le a^-$ and $a^+ \le b^+$.

It is easily to see that if we define

$$\bar{a} \cdot \bar{b} = \{x \cdot y | x \in \bar{a}, y \in \bar{b}\}$$

for $\bar{a}, \bar{b} \in I(R^+)$, then

$$\bar{a} \cdot \bar{b} = [a^- \cdot b^-, a^+ \cdot b^+]$$

and that if $d_H : I(R^+) \times I(R^+) \to [0, \infty)$ is a Hausdorff metric, then

$$d_H(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Definition 3.2 ([1,2,3,6]) (1) An interval-valued function $\bar{f}$ is said to be measurable if for each open set $O \subset R^+$,

$\bar{f}^{-1}(O)=\{x\in X|\bar{f}(x)\cap O\neq\varnothing\}\in\Omega$.

(2) An interval-valued function $\bar{f}$ is said to be finite if $\|\bar{f}\|=\sup_{r\in\bar{f}(x)}|r|<\infty$

We denote $\mathbb{IF}$ by the class of all finite measurable interval-valued functions $\bar{f}=[f^-,f^+]:X\rightarrow I(R^+)\setminus\{\varnothing\}$ on X.

Definition 3.3 Let $\bar{f}\in\mathbb{IF}$ and $\{\bar{f}_n\}\subset\mathbb{IF}$. $\{\bar{f}_n\}$ is said to d_H-converge to $\bar{f}$ almost everywhere (resp. pseudo-almost everywhere) on A if there is a subset $E\subset A$ such that $\mu(E)=0$ (resp. $\mu(A-E)=\mu(A)$) and $\bar{f}_n$ d_H-converges to $\bar{f}$ on $A-E$, that is,

$$\lim_{n\rightarrow\infty}d_H(\overline{f_n}(x),\bar{f}(x))=0,\text{ for all } x\in A-E.$$

Definition 3.4 Let $\bar{f}\in\mathbb{IF}$ and $\{\bar{f}_n\}\subset\mathbb{IF}$. $\{\bar{f}_n\}$ is said to d_H-converge to $\bar{f}$ in measure μ(resp. pseudo-in measure μ) on A if for any $\epsilon>0$,

$$\lim_{n\rightarrow\infty}\mu(\{x:d_H(\overline{f_n}(x),\bar{f}(x))\geq\epsilon\}\cap A)=0$$

(resp.

$$\lim_{n\rightarrow\infty}\mu(\{x:d_H(\overline{f_n}(x),\bar{f}(x))<\epsilon\}\cap A)=\mu(A)).$$

Definition 3.5 ([3]) (1) Let $A\in\Omega$. The Choquet integral of an interval-valued $\bar{f}$ on A is defined by

$$(C)\int_A\bar{f}d\mu=\{(C)\int_A fd\mu|f\in S(\bar{f})\}$$

where $S(\bar{f})$ is the family of measurable selections of $\bar{f}$.

(2) $\bar{f}$ is said to be c-integrable if

$$(C)\int\bar{f}d\mu\neq\varnothing.$$

(3) $\bar{f}$ is said to be Choquet integrably bounded if there is a c-integrable function g such that

$$\|\bar{f}\|=\sup_{r\in\bar{f}(x)}|r|\leq g(x),$$

for all $x\in X$.

Theorem 3.6 ([6]) If a fuzzy measure μ is continuous and an interval-valued function $\bar{f}=[f^-,f^+]$ is Choquet integrably bounded, then

$$(C)\int\bar{f}d\mu=[(C)\int f^-d\mu,(C)\int f^+d\mu].$$

We denote $\mathbb{IF}^*$ by the class of all Choquet integrably bounded interval-valued functions in $\mathbb{IF}$.

Lemma 3.7 Let $\bar{f}=[f^-,f^+]\in\mathbb{IF}$ and $\{\bar{f}_n\}=\{[f_n^-,f_n^+]\}\subset\mathbb{IF}$.

(1) $\{\bar{f}_n\}$ d_H-converge to $\bar{f}$ almost everywhere (resp. pseudo-almost everywhere) on A if and only if $\{f_n^-\}$is said to converge to f^- almost everywhere (resp. pseudo-almost everywhere) on A and $\{f_n^+\}$is said to converge to f^+ almost everywhere (resp. pseudo-almost everywhere) on A.

(2) $\{\bar{f}_n\}$ is said to d_H-converge to $\bar{f}$ in measure μ(resp. pseudo-in measure μ) on A if and only if $\{f_n^-\}$is said to converge to f^- in measure μ(resp. pseudo-in measure μ) on A and $\{f_n^+\}$is said to converge to f^+ in measure μ(resp. pseudo-in measure μ) on A.

we discuss the equivalence among the Lebesgue's theorems, the monotone convergence theorems of interval-valued fuzzy integrals with respect to a monotone set function.

Theorem 3.8 The following three statements are equivalent.

(1) μ is continuous from below;

(2) for any $A\in\Omega$, $\bar{f}\in\mathbb{IF}$ and $\{\bar{f}_n\}\subset\mathbb{IF}$, $\{\bar{f}_n\}$ d_H-converge to $\bar{f}$ pseudo-almost everywhere on A imply $\{\bar{f}_n\}$ d_H-converge to $\bar{f}$ pseudo-in measure μ on A;

(3) for any $A \in \Omega$, $\bar{f} \in \mathbb{IF}$ and $\{\bar{f}_n\} \subset \mathbb{IF}$, $\bar{f}_n \nearrow \bar{f}$ on A imply

$$d_H-\lim_{n\to\infty}(S)\int_A \bar{f}_n d\mu=(S)\int_A \bar{f} d\mu.$$

Theorem 3.9 The following three statements are equivalent.

(1) μ is null additive and continuous from below;

(2) for any $A \in \Omega$, $\bar{f} \in \mathbb{IF}$ and $\{\bar{f}_n\} \subset \mathbb{IF}$, $\{\bar{f}_n\}$ d_H-converge to $\bar{f}$ pseudo-almost everywhere on A imply $\{\bar{f}_n\}$ d_H-converge to $\bar{f}$ pseudo-in measure μ on A;

(3) for any $A \in \Omega$, $\bar{f} \in \mathbb{IF}$ and $\{\bar{f}_n\} \subset \mathbb{IF}$, $\bar{f}_n \nearrow \bar{f}$ on A imply

$$d_H-\lim_{n\to\infty}(S)\int_A \bar{f}_n d\mu=(S)\int_A \bar{f} d\mu.$$

Now, we find some sufficient condition that the monotone convergence theorem of interval-valued Choquet integrals with respect to a monotone set function holds.

Theorem 3.10 If μ is continuous and for any $A \in \Omega$, $\bar{f} \in \mathbb{IF}^*$ and $\{\bar{f}_n\} \subset \mathbb{IF}^*$, $\bar{f}_n \nearrow \bar{f}$ on A, then

$$d_H-\lim_{n\to\infty}(C)\int_A \bar{f}_n d\mu=(C)\int_A \bar{f} d\mu.$$

Finally, we obtain some properties of interval-valued Choquet integrals and interval-valued fuzzy integrals with respect to a monotone set function.

Theorem 3.11 Let μ be continuous. Then the following two statements are equivalent.

(1) For any $A \in \Omega$, $f \in \mathbb{F}^*$ and $\{f_n\} \subset \mathbb{F}^*$, $f_n \nearrow f$ on A, then

$$\lim_{n\to\infty}(C)\int_A f_n d\mu=(C)\int_A f d\mu;$$

(2) For any $A \in \Omega$, $\bar{f} \in \mathbb{IF}^*$ and $\{\bar{f}_n\} \subset \mathbb{IF}^*$, $\bar{f}_n \nearrow \bar{f}$ on A, then

$$d_H-\lim_{n\to\infty}(C)\int_A \bar{f}_n d\mu=(C)\int_A \bar{f} d\mu.$$

Theorem 3.12 Let μ be continuous from below. Then the following two statements are equivalent.

(1) For any $A \in \Omega$, $f \in \mathbb{F}$ and $\{f_n\} \subset \mathbb{F}$, $f_n \nearrow f$ on A, then

$$\lim_{n\to\infty}(S)\int_A f_n d\mu=(S)\int_A f d\mu;$$

(2) For any $A \in \Omega$, $\bar{f} \in \mathbb{IF}$ and $\{\bar{f}_n\} \subset \mathbb{IF}$, $\bar{f}_n \nearrow \bar{f}$ on A, then

$$d_H-\lim_{n\to\infty}(S)\int_A \bar{f}_n d\mu=(S)\int_A \bar{f} d\mu.$$

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