시간 지연을 포함한 네트워크 시스템의 안정도 분석

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Stability Analysis of Network Systems with Time delay

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Abstract - This paper presents a stability analysis of network systems with time delay. Time delay problem frequently occurs in network systems. Since it makes network systems unstable and unpredictable, an optimal controller is necessary to network systems. We prove the asymptotical stability of time delayed network systems using LMI optimization method and appropriate Lyapunov-Krasovskii functionals. Simulations show the effectiveness of the method.

1. Introduction

Congestion control of TCP network is an essential problem in the Internet [1,2]. The active queue management(AQM) was introduced for this end-to-end congestion control in the Internet. AQM primarily responds to network congestion when a queue begins to increase and then maintains a queue size at a predefined level in the router. RED algorithm has been proposed for AQM scheme. It is the first practical and most well known method. PI algorithm also has been proposed to AQM scheme, which is a control-theoretic AQM approach. Furthermore, REM, PID, neural network were used for AQM. But, these methods treated time delay as unimportant.
They problems frequently occur in many systems due to the intentional designing as well as the internal characteristic of systems. Time delays may make systems unstable and they are occasionally a cause of poor performance. Therefore, stability analysis of time delayed systems has been an important issue in recent years [3,4].
In this paper, we propose a stability analysis of network systems with time delay. Using the LMI method and Lyapunov-Krasovskii functionals, the stability of time-delayed network systems is analyzed.

2. Problem Formulations

A dynamic model of TCP behavior was developed using fluid-flow and stochastic differential equation analysis in [5]. A simplified version of this model, which does not include the TCP timeout mechanism is given as follows:

$$\dot{W}(t) = \frac{1}{R(t)} - \frac{W(t)\,W(t-R(t))}{2R(t-R(t))}p(t-R(t)),$$
$$\dot{q}(t) = \frac{W(t)}{R(t)}N(t) - C, \tag{1}$$

where W denotes TCP window size, q is a queue length at a router, $R(t)$ is a round trip time (RTT), calculated by $q(t)/C+T_p$. Here C is a link capacity, T_p is a propagation delay, N is the number of TCP sessions(load factor) and p is the probability of a packet loss.
The system (1) is linearized as follow [6]:

$$\delta\dot{W}(t) = -\frac{N}{R_0 C}(\delta W(t)+\delta W(t-R_0)) - \frac{1}{R_0 C}(\delta q(t)+\delta q(t-R_0)) - \frac{R_0 C^2}{2N^2}\delta p(t-R_0)$$
$$\delta\dot{q}(t) = \frac{N}{R_0}\delta W(t) - \frac{1}{R_0}\delta q(t) \tag{2}$$

where, $\delta W = W - W_0, \delta q = q - q_0, \delta p = p - p_0$, in which W_0, q_o, p_0 are the operating points. However, in fact, the linearized system (2) only corresponding with the nonlinear system (1) at the operating point, but does not equivalent to the nonlinear system at other points. Therefore, to overcome this problem, the time-delayed network system (1) represents as follows:

$$\dot{x}(t) = Ax(t) + A_1 x(t-h_1(t)) + Bu(t) + B_1 u(t-h_2(t)) + f(x(t), x(t-h_1(t), u(t), u(t-h_2(t))),\ \ t>0,$$
$$x(t) = \phi(t), \qquad t\in[-h,0], \tag{3}$$

where, $x = [\delta W\ \delta q]^T, \quad u = \delta p, \quad h_1(t) = h_2(t) = R(t)$,

$$A = \begin{bmatrix} -\frac{N}{R_0^2 C} & -\frac{1}{R_0^2 C} \\ \frac{N}{R_0} & -\frac{1}{R_0} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -\frac{N}{R_0^2 C} & -\frac{1}{R_0^2 C} \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -\frac{R_0^2 C}{2N^2} \\ 0 \end{bmatrix} \tag{4}$$

and $f(\bullet)$ denotes an uncertain nonlinear function.

Assumption 1. The uncertain nonlinear function $f(\bullet)$ satisfies

$$f^T f \le \|H_0 x(t)\|^2 + \|H_1 x(t-h_1(t))\|^2 + \|H_2 x(t-h_2(t))\|^2 \tag{5}$$

where, H_0, H_1, H_2 are known constant matrices.
And the controller has following form:

$$u(t) = (K+\Delta K)x(t) \tag{6}$$

where, $\Delta K = DF(t)E$ represents the perturbation matrix.
Associated with the time-delayed network system (3) and the controller (6), we consider the cost function as follows:

$$J = \int_0^\infty x^T(t)Q_1 x(t) + [u^T(t)K]Q_2[K^T u(t)]dt. \tag{7}$$

3. Main Results

Applying the state feedback controller (6) to network system (3), the closed loop system is represented as follows:

$$\dot{x}(t) = [A + B(K+\Delta K) + A_1 + B_1(K+\Delta K)]x(t) - A_1\int_{t-h_1(t)}^{t}\dot{x}(s)ds - B_1(K+\Delta K)\int_{t-h_2(t)}^{t}\dot{x}(s)ds + f,\ \ t>0,$$
$$x(t) = \phi(t),\ \ t\in[-h,0], \tag{8}$$

Then, we have the following theorem.

Theorem 1. Consider the network systems with time-delay (3) and the cost function (7). If there exist a controller of the form (6), positive definite matrices P, S_i, R, Z_i, any matrices $Y_i, i=1,2$ and a positive scalar δ_1 which satisfy the following inequalities:

$$\Gamma = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \star & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \star & \star & \Psi_{33} & \Psi_{34} \\ \star & \star & \star & \Psi_{44} \end{bmatrix} < 0, \tag{9}$$

and

$$\begin{bmatrix} Z_i & Y_i \\ \star & R_i \end{bmatrix} \ge 0,\ i=1,2, \tag{10}$$

where

$$
\begin{aligned}
\Psi_{11} &= (A+B(K+\Delta K))^T P + P(A+B(K+\Delta K)) \\
&\quad + (A+B(K+\Delta K))^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)(A+B(K+\Delta K)) \\
&\quad + \sum_{i=1}^{2}\left\{Y_i + Y_i^T + \overline{h}_i Z_i\right\} + \delta_1^{-1} H_0^T H_0 + (S_1 + S_2), \\
\Psi_{12} &= (A+B(K+\Delta K))^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)A_d + PA_d - Y_1, \\
\Psi_{13} &= (A+B(K+\Delta K))^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)B_d(K+\Delta K) \\
&\quad + PB_d(K+\Delta K) - Y_2, \\
\Psi_{22} &= A_d^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)A_d + \delta_1^{-1} H_1^T H_1 + (1-d_1)S_1, \\
\Psi_{23} &= A_d^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)B_d(K+\Delta K), \\
\Psi_{24} &= A_d^T(\overline{h}_1 R_1 + \overline{h}_2 R_2), \\
\Psi_{33} &= (K+\Delta K)^T B_d^T(\overline{h}_1 R_1 + \overline{h}_2 R_2)B_d(K+\Delta K) \\
&\quad + \delta_1^{-1} H_2^T H_2 - (1-d_2)S_2, \\
\Psi_{34} &= (K+\Delta K)^T B_d^T(\overline{h}_1 R_1 + \overline{h}_2 R_2), \\
\Psi_{44} &= (\overline{h}_1 R_1 + \overline{h}_2 R_2) - \delta_1^{-1} I,
\end{aligned} \tag{11}
$$

then the close-loop system (3) is asymptotically stable and the cost function (7) satisfies the following upper bound:

$$
J \le x^T(0)Px(0) + \sum_{i=1}^{2}\int_{-h_i(t)}^{0}\int_{\theta}^{0}\dot{x}(s)R_i\dot{x}(s)ds d\theta
$$

$$
+ \sum_{i=1}^{2}\int_{h_i(t)}^{0} x^T(p)S_i x(p)dp. \tag{12}
$$

Proof. Choose appropriate Lyapunov-Krasovskii functionals as follows:

$$
V(t) = V_1(t) + V_2(t) + V_3(t) \tag{13}
$$

where

$$
\begin{aligned}
V_1(t) &= x^T(t)Px(t) \\
V_2(t) &= \sum_{i=1}^{2}\left\{\int_{-h_i(t)}^{0}\int_{-t+\theta}^{t}\dot{x}^T(s)R_i\dot{x}^T(s)ds d\theta\right\} \\
&\quad - \sum_{i=1}^{2}\left\{d_i\int_{0}^{t}\int_{t-h_i(t)}^{t}\dot{x}^T(s)R_i\dot{x}^T(s)ds dt\right\} \\
V_3(t) &= \sum_{i=1}^{2}\left\{\int_{-t+h_i(t)}^{t} x^T(p)S_i x(p)dp\right\}.
\end{aligned} \tag{14}
$$

Taking the time derivative of $V(t)$ along the trajectories of the system (3) and (5), we obtain

$$
\dot{V}(t) \le \begin{bmatrix} x(t) \\ x(t-h_1(t)) \\ x(t-h_2(t)) \\ f \end{bmatrix}^T \Gamma \begin{bmatrix} x(t) \\ x(t-h_1(t)) \\ x(t-h_2(t)) \\ f \end{bmatrix}
$$

$$
- x^T(t)\left[Q_1 + (K+\Delta K)^T K Q_2 K^T(K+\Delta K)\right]x(t) \tag{15}
$$

Then, the matrix inequality (9) implies that

$$
\dot{V}(t) < -x^T(t)\left[Q_1 + (K+\Delta K)^T K Q_2 K^T(K+\Delta K)\right]x(t) < 0. \tag{16}
$$

Since Q_1 and Q_2 are given symmetric positive definite matrices, the system (3) is asymptotically stable by Lyapunov stability theory. ▮

Theorem 2. Consider the time delayed network systems (3) and the cost function (7). If there exist scalars δ_1, δ_2, and positive definite matrices L, U_i, Z_i, R_i, any matrices $W, M, V, J, Y_i, i=1,2$ satisfying the following inequalities:

$$
\Lambda = \begin{bmatrix}
\Omega_{11} & C_1 & G^T & G^T & \delta_2 BD & C_4 & C_5 \\
\star & D_1 & C_2 & C_2 & 0 & 0 & 0 \\
\star & \star & -\overline{h_1^{-1}}R_1^{-1} & 0 & 0 & 0 & 0 \\
\star & \star & \star & -\overline{h_2^{-1}}R_2^{-1} & C_3 & 0 & 0 \\
\star & \star & \star & \star & -\delta_2 I & 0 & 0 \\
\star & \star & \star & \star & \star & -\delta_1 I & 0 \\
\star & \star & \star & \star & \star & \star & D_2
\end{bmatrix} < 0 \tag{17}
$$

where

$$
\begin{aligned}
&\Omega_{11} = G + G^T, \ \ G = AL + BV, \ \ L = P^{-1}, \\
&V = KP^{-1}, \ \ W = K^T KP^{-1}, \ U_1 = S_i^{-1}, \ \ i = 1,2, \\
&C_1 = \left[(A_d U_1)\,(B_d K U_2)\,\delta_1 I\right], \\
&C_2 = \left[(A_d U_1), (B_d K U_2), \delta_1 I\right]^T, \\
&C_3 = \left[\delta_2 BD\ \delta_2 BD\ \delta_2 JD\right]^T, \\
&C_4 = \left[(H_0 L)\,(H_1 U_1)\,(H_2 U_2)\right]^T, \\
&C_5 = \left[W^T\,(EL)^T\,L^T \cdots L^T\right], \\
&D_1 = diag\left\{(d_1-1)U_1, (d_2-1)U_2, -\delta_1 I\right\}, \\
&D_2 = diag\left\{-R^{-1}, -\delta_2 I, -\overline{h_1^{-1}}Z_1^{-1}, -\overline{h_2^{-1}}Z_2^{-1}, -U_1, -U_2\right\},
\end{aligned} \tag{18}
$$

then the controller is given by

$$
u(t) = VL^{-1}x(t). \tag{19}
$$

Proof. Substituting the perturbation matrix ΔK into (9) and multiplying the inequality Γ by $diag\left\{(P^{-1})^T, (S_1^{-1})^T, (S_2^{-1})^T, (\delta_1 I)^T\right\}$ and $diag\left\{(P^{-1}), (S_1^{-1}), (S_2^{-1}), (\delta_1 I)\right\}$ on the left and right side, respectively. the obtained inequalities are equivalent to (17). ▮

4. Simulations

Consider the time-delayed network system as follows:

$$
A = \begin{bmatrix} -\frac{N}{R_0^2 C} & -\frac{1}{R_0^2 C} \\ \frac{N}{R_0} & -\frac{1}{R_0} \end{bmatrix}, \ \ A_1 = \begin{bmatrix} -\frac{N}{R_0^2 C} & -\frac{1}{R_0^2 C} \\ 0 & 0 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \ B_1 = \begin{bmatrix} -\frac{R_0^2 C}{2N^2} \\ 0 \end{bmatrix}, H_0 = H_1 = H_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
$$

$$
D = 0.2, E = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}^T, h_1 = h_2 = 0.3, d_1 = d_2 = 0.2
$$

$$
x(t) = \left[e^{-0.5t}\ e^{-0.5t}\right]^T, \ h \le t \le 0. \tag{20}
$$

where, $C = 3750$(packet/sec), $R_0 = 0.246$(sec), $N = 960$. And the weighting matrices in the cost function (7) are $Q_1 = I$ and $Q_2 = 0.5I$, respectively.

Using the LMI toolbox, we obtain the guarantee cost controller as follows:

$$
u(t) = [-43.1532\ -18.1064]x(t). \tag{21}
$$

5. Conclusions

We have presented a stability analysis of time-delayed network systems. LMI optimization method and appropriate Lyapunov-Krasovskii functionals have showed the stability of the time-delayed network systems. Furthermore, an optimal controller gain is obtained by solving the feasible LMI solution, and the cost function is guaranteed to be less than specific upper bounded level regardless of the network systems performance deteriorated by the time delay. Simulations show that the time-delayed network systems are asymptotically stable.

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