韓國數學教育學會
<全國數學教育研究大會 프로시딩>
제 37회. 2006. 10. 1-19

Korea Soc. Math. Ed.
Proceedings of National Meeting of Math. Ed.
The 37th. Oct. 2006. 1-19

A New Curriculum for Structural Understanding of Algebra

David Kirshner (Louisiana State University)

A New Curriculum for Structural Understanding of Algebra

Algebra is tough, or at least that's the reputation its garnered for itself. You need to be smart to understand algebra and master its arcane rules and methods. This perception is supported by decades of studies of persistent error patterns, or "mal-rules" (Sleeman, 1986) that seem to indicate students have considerable difficulty understanding algebraic rules (Table 1). Still, research about learning is developed and interpreted within theoretical traditions based on assumptions about the nature of human cognition. If those assumptions are called into question, research results can be reinterpreted.

But how, you may ask, could students' persistent errors possibly indicate anything other than that algebra is difficult to understand? Well hold on to your safety harness as we peer down on thirty years of algebra error research from the heights of Western philosophy. Though, quite bumpy and bruising to traditional assumptions about learning, the ride, I assure you, is well worth it, for we parachute down amidst new vistas of possibility for students' algebraic accomplishment.

Table 1
Mal-rules and Correct Rules

Mal-rules	Correct Rules
$(a+b)^c = a^c + b^c$	$(ab)^c = a^c b^c$
$\sqrt[c]{a+b} = \sqrt[c]{a} + \sqrt[c]{b}$	$\sqrt[c]{ab} = \sqrt[c]{a}\sqrt[c]{b}$
$a^{mn} = a^m a^n$	$a^{m+n} = a^m a^n$
$a^{m+n} = a^m + a^n$	$a(m+n) = am + an$
$\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$	$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$
$\frac{a+x}{b+x} = \frac{a}{b}$	$\frac{ax}{bx} = \frac{a}{b}$

Traditional algebra classrooms are familiar places to all of us who are interested in mathematics education at the secondary school. Textbooks present rules for manipulating expressions and equations. Teachers explain and demonstrate the rules. Students practice solving routine application problems according to the models they've been shown. To the extent students can reliably demonstrate required skills we conclude our explanations and demonstrations have been understood, at least at a rudimentary level. For students who become ensnared by error patterns like those in Table 1, we lament that our discussion of rules may have been too abstract-the students have not comprehended, or have not worked hard enough to consolidate their understanding. In the United States, we find only 40% of Grade 12 students solve moderately complex symbol manipulation problems without error (Blume & Heckman, 1997), so we become discouraged about even presenting the abstract face of algebra to our students.

This familiar view of algebra skills as reflecting understanding of rules and procedures presented in the curriculum is consistent with cognitive assumptions that undergird 3 decades of algebra learning research within the Information Processing (IP) tradition of cognitive psychology. For instance, Carry, Lewis, and Bernar's (1980) cognitive analysis started with "the legal moves of the algebra game" (p. 2). More explicitly, Matz (1980)

> idealizes an individual's problem-solving behavior as a process employing two components. The first component, the knowledge presumed to precede a new problem, usually takes the form of a rule a student has extracted from a prototype or gotten directly from a textbook. For the most part these are basic rules (such as the distributive law, the cancellation rule, the procedure for solving factorable polynomials using the zero product principle) that form the core of the conventional textbook content of algebra. (p. 95)

The second component of her framework explored how error patterns arise through (mis)application of "*extrapolation techniques* that specify ways to bridge the gap between known rules and unfamiliar problems" (Matz, 1980, p. 95).

The idea that students' competence in algebraic skills depends upon mastering and applying explicitly given rules is comfortable precisely because it fits so well with the Cartesian dualist philosophical assumptions of Western thought (Brooks, 1991; Clancey, 1999; Dupuy, 2000; Estep, 2003). Cartesian dualism (yes, owing to René Descartes of Cartesian Coordinates fame) is the "principle of the separation of mind and matter and mind and body. The mind, according to

Descartes, was a 'thinking thing', and an immaterial substance. ... the essence of himself, the part that doubts, believes, hopes, and so on. The body is a material substance" (Wikipedia, 2005, http://en.wikipedia.org/wiki/Cartesian_dualism). Descartes believed that mind acts upon body to produce directed actions and behaviors. But this interaction between separate "substances" has always been taken as a kind of philosophical conundrum. However, Information Processing theory, based on the computational metaphor of mind as computer, solves "the metaphysical problem of mind interacting with matter" (Haugland, 1985, p. 2). For IP theories specify control structures through which skills are executed based on a centralized serial script. In algebra, that script is taken to include the rules of algebra as presented in the curriculum. So learning algebra must involve grasping the rules and incorporating them into mental scripts (as in Matz's, 1980, model). Indeed, so embedded are we in the dualist ideas of our culture, it is hard to even make sense of the proposal that students might acquire algebra skills without even engaging with the meaning of the rules presented in the curriculum, let alone understanding the rules.

Despite the dominance of dualism, our intuitions as mathematics educators and researchers often point us in other directions. Consider the mal-rules presented above. What is so confounding about such errors is their superficial character. Rather than reflecting mis*understanding* of the *meaning* of correct algebra rules, they seem to indicate nothing more substantial than mis*perception* of the visual *forms* of the correct rules. It is in this sense that Thompson (1989) spoke of algebra students as "prone to pushing symbols without engaging their brains" (p. 138). Somehow students seem to be riding the visual forms of algebraic notations without engaging with the meaning of the rules. For instance, in his landmark study Erlwanger (1973) observed:

> One may be tempted to treat this kind of talk as evidence of an algebraic concept of commutativity. But, in view of the whole picture of Benny's concept of rules, it appears more likely that it involves less awareness of algebraic operations than it does awareness of patterns on the printed page. (Note to p. 19)

Of course, these exclamations of frustration with "mindless" manipulation of algebraic symbols involve students who are struggling in their mastery of the requisite skills. But what of the successful students? Isn't it possible that the failing students are those who have given up on understanding and reverted to mindless manipulation of symbols, whereas the successful students operate from understanding? In two studies, I set out to explore the basis for successful algebra manipulation skills, only to discover that visual pattern matching underlies much of the successful work our students produce.

The first study examined students' parsing knowledge. Supposing a student can successfully parse an expression like $3x^2$ [as $3(x^2)$ rather than $(3x)^2$]. From a dualist perspective we assume the student has explicit knowledge of order of operations rules (e.g., exponentiation has precedence over multiplication). But Kirshner (1989) demonstrated such competencies often are linked to the visual characteristics of the symbol system. What the student really knows at an implicit level is something more like "diagonal connections precede horizontal connections." When such tasks were presented in an alternative notation, 3M×E2, that preserves the declarative information about the operations ("M"= multiplication, and "E"=exponentiation), students' ability to make the correct parsing decision was compromised. For many students, parsing knowledge was inextricably linked to the visual cues of ordinary notation. They didn't "know" the rules in an intellectual sense, rather they were adept at maneuvering within the visual space of the printed symbols.

Table 2
Visually Salient and Non Visually-salient Transformational Rules

$x(y+z)=xy+xz$	$x^2-y^2=(x-y)(x+y)$
$(xy)^2=x^2y^2$	$(x+y)^2=x^2+2xy+y^2$
$(x^y)^z=x^{yz}$	$x^3+y^3=(x+y)(x^2-xy+y^2)$
$\frac{w}{x}\frac{y}{z}=\frac{wy}{xz}$	$\frac{w}{x}+\frac{y}{z}=\frac{wz+xy}{xz}$
$\frac{xy}{xz}=\frac{y}{z}$	$\frac{w}{x}\div\frac{y}{z}=\frac{wz}{xy}$

The second study examined the nature of students' engagement with *transformational rules* such as those presented in Table 2. Take a close look at the two columns of rules. Note that some rules, like those in the left column, have a certain quality of *visual salience* that makes the left-hand and righthand sides of the equation look naturally related to one another. The quality of visual salience is easy to recognize but difficult to define. Partly it involves symbolic and spatial elements repeated from the left-hand to the right-hand side of the equation. But whatever the causes, its effects are similar to an animation sequence in which distinct visual frames are perceived as ongoing instances of a single scene-think of an animated cartoon made up of separate pictures, but perceived as a single evolving visual scene (Kirshner & Awtry, 2004). So for visually salient rules the observer doesn't have to stop to process conscious connections between two separate sides of the equation, for they are not perceived as separate entities, but as a single entity transformed over time.

What we found in our study is that students learn visually salient rules more easily than "clunky" rules that are not visually salient, but at a more superficial level. That is, they could recognize correct applications of the visually salient rules more easily than the clunky rules, but they also were more likely to overgeneralize visually salient rules to cases where the rule didn't actually apply, but "looked" right. This is not surprising, as the visually salient rules are the spawning ground for virtually all of the mal-rules that beset students' efforts at mastering algebra symbol skills (look again at Table 1). But what is noteworthy is that our study involved novices-grade 7 students who had not previously been introduced to algebra rules. We taught these students a mixed set of visually salient and clunky rules in a strict didactic fashion, not differentiating in our instruction between rule types. Yet students engaged spontaneously with the visual surface structure of the notations. Reliance of visual cues is not a pathological adaptation of students who are failing, but a norm for how students engage cognitively with the algebra symbol system from the very start.

Although dualist ideas dominate in Western philosophy and in psychology, there is considerable scholarly debate about the adequacy of mentalist models of cognitive competencies. I want to take a few moments to introduce connectionist psychology, an approach to modeling cognitive skills that has arisen in recent years to challenge the hegemony of rule based models like those developed in IP theories (see Bereiter, 1991; Dreyfus, 2002; Gee, 1992). My purpose, here, is to help establish as a real possibility that our cultural commonsense about learning, rooted in dualist philosophy, may be wrong. Skillful performance in a domain like algebra doesn't have to be based on learning "the legal moves of the algebra game" (Carry, Lewis, & Bernard, 1980, p. 2), or on "a rule a student has extracted from a prototype or gotten directly from a textbook" (Matz, 1980, p. 95). Cognitive skills can be something quite different.

Connectionism represents cognition as a spreading of activation among nodes in a system of mutually interacting nodes-rather like neurons interacting within the brain. Typically connectionist systems include input nodes corresponding to features of the domain to be mastered and output nodes related to actions that can be taken or decisions that can be reached, as well as hidden units that intermediate between input and output nodes. But whereas an Information Processing system would have to spell out in explicit rules how inputs and outputs are linked, connectionist systems work differently. Nodes are interconnected through links that carry excitory and/or inhibitory signals (Haberlandt, 1997). When a certain threshold of activation is reached, the node sends signals to those other nodes to which it is connected. In this way, connectionism pictures cognition as involving parallel and distributed processing, rather than centralized, serial script following.

In a connectionist system, the links between nodes are weighted, meaning that a link may impede the signal passing along it, thereby reducing its effect; or it may allow the full strength of the signal to reach its destination (Bereiter, 1991). Learning is modeled as feedback loops that gradually adjust the connection weights between nodes according to the effectiveness of the system's previous actions (Lloyd, 1989; Rumelhart, Hinton, & Williams, 1986). These gradual adjustments of the connection weights lead to a gradual improvement of performance as the system tends toward a steady state. In short, connectionism models cognitive skills as weighted correlations among a large number of input, output, and intermediate nodes. No centralized rule based program runs the show.

Part of the impetus for pursuing connectionist alternatives is that IP models "are cumbersome and brittle; they tend to break down when the stimulus conditions are poorly specified" (for example in visual discrimination tasks like recognizing faces from different angles, even faces that may have altered with age). "Connectionist models are well suited for just such situations" (Haberlandt, 1997, p. 159). So, if we forget (as students are wont to do) about rules as having meaning, we can imagine that connectionist like processes lead to skill development based only on connections among visual features of the symbolic display. This doesn't mean that discussion of rules plays no role in learning of algebra. For talk about rules can focus the cognitive apparatus on input and output features that productively are incorporated into the connectionist system as input and output nodes: "Rules, thus, may play an important role as *knowledge that enters into computations*, but this is a fundamentally different role from the one traditionally conceived by philosophers and cognitive scientists, where *rules constitute the computational algorithms themselves*" (Bereiter, 1991, p. 14). Or to put it more plainly, rules can help us focus on critical elements, but this doesn't have anything to do with understanding rules and using them as explicit guides for action: "from a connectionist standpoint such rule-based models are fictions" (Bereiter & Scardamalia, 1996, p. 509).

This connectionist view of cognition is challenging to all of us rooted, as we are, in the dualist perspectives of our culture. But connectionism does constitute a coherent alternative to the IP approach-computer simulations have been developed that perform effectively in structured domains, without guidance of rules.

Let's return to the traditional algebra classroom sketched in the opening paragraphs, but now reviewed through non-dualist, connectionist lenses. Textbooks and teachers present explicit rules and procedures for solving algebra problems. Students practice such problems, and eventually get better at them. But these two classroom events are largely independent of one another. Developing expertise is not based on "understanding" and applying the rules presented in the curriculum.

Rather, through practice, students gradually refine the pattern matching nets that organize activity within the visual space of the printed symbols. With enough attention and perseverance, a minority of students eventually do overcome the tendency to produce mal-rules. But these mal-rules are not "misunderstandings," they are visual overgeneralizations, and overcoming them just signals that the pattern matching nets have become sufficiently refined-not that "understanding" has been achieved. Because we offer reasoned explanations to our students about algebraic rules and processes, we create an illusion for ourselves that algebra is an intellectual study. But what we're really doing is leaving our students to flounder mindlessly in a sea of visual signals.

The Possibility for Meaningful Manipulation of Algebraic Symbols

My goal for the rest of this paper is to outline a curricular approach aimed at making algebraic symbol manipulation meaningful for students. But first I need to deal with what may strike the reader as a bait-and-switch strategy. For I've spent the first half of the chapter arguing that cognition is parallel and distributed, that rule-based models are "fictions," that algebra skills are a correlation of surface visual features of the notation; now I'm trying to sell you on meaningful algebraic symbol manipulation. What Happened to the Non-dualist Perspective!

If connectionism views cognition as a spreading of activation among nodes in a system of mutually interacting nodes, what are we to make of rationality, logic, reasoning? Are these kinds of centralized, serial processes now banished from mental life? The answer, in a nutshell, is that rationality, logic, reasoning are lifted out of the core inner mechanisms of cognition (where dualist philosophers have supposed they structure cognitive scripts that govern performance), and spun off as discursive practices within the social world (Kirshner, 2001; Bereiter, 1991, Gee 1992):

> Rationality ... originates in this essentially social process of justification. What we call logical reasoning, and attribute to the workings of the individual mind, is actually a public reconstruction meant to legitimate a conclusion by showing that it can be derived by procedures recognized as valid. (Bereiter, 1991, p. 14)

What this means is that the goal we mathematics educators hold precious-that our students will become reasoners about their mathematical work-is dramatically transformed when we adopt a nondualist perspective. Instead of thinking that rationality, logic, and reasoning are built into the tasks we assign to students, we need to enact these cognitive attributes within the social milieu of the classroom. Rationality resides in the justification of algebra problem solving, not in the solving of algebra problems.

What has most seriously impeded the development of an algebra curriculum to support explicit reasoning about rules and procedures is the dualist assumption that solving algebra problems already involves students in reasoning. We consistently misconstrue students' correct algebraic work as indicating explicit knowledge, and hence never bother to do our own analysis of what''s really involved in step-by-step justification of algebraic procedures.

Let me illustrate this point by describing a teaching episode I observed not too long ago. The teacher, of several years experience, was working with her class on solving the equation $-2+4n+9=20$. She had just written on the board a solution method offered by one student who added 2 to both sides, then subtracted 9 from both sides, and finally divided both sides by 4. The method is correct (though collecting like terms first would have been more efficient). Here is the first step of this student's work:

$$-2+4n+9=20 \quad \text{so} \quad \begin{array}{l}-2+4n+9=20\\ +2 \qquad\qquad\quad +2\end{array} \quad \text{so} \quad 4n+9=22$$

Next the teacher transcribed onto the board an incorrect method offered by another student, beginning with this step:

$$-2+4n+9=20 \quad \text{so} \quad -2+\frac{4n}{4}+9=\frac{20}{4} \quad \text{so} \quad -2+n+9=5$$

The teacher corrected the student by pointing out that you have to do the same thing to the <u>whole side</u> of an equation, not just to a single term. This is a correct explanation. Clearly the teacher knew what she was talking about. And she is to be commended on developing a classroom process in which students have the opportunity to present their ideas and solutions. But is the discourse of this classroom really adequate to facilitate meaning making on the part of the students?

Let's look at this episode from the point of view of the students. Note that the correct step and the incorrect step shown above look similar. In both cases, it looks like an operator is being applied only to a single term on the left-hand side of the equation. What we would need to do to make the actual structure of the first solution explicit would be something like the following:

$-2+4n+9$	$=20$	
$(-2+4n+9)+2$	$=20+2$	law of equations
$2+(-2+4n+9)$	$=20+2$	commutative law for addition
$(2+-2)+(4n+9)$	$=20+2$	associative law for addition

But, like the teacher I observed, we mostly don't do this, because we are satisfied that somehow the student getting the correct answer already "understands" what is going on. But in all likelihood neither the successful nor the unsuccessful student understands the application of associative and commutative laws implicit in such derivations. Both are merely aping visual formats for solving equations they've seen modeled in the classroom. And the teacher's admonition-you have to do the same thing to the whole side-though correct, communicates nothing to the students, because the teacher has never bothered to think through the implications of what "doing the same thing to the whole side" means in structural terms. This is what can make teaching algebra such a frustrating experience as we hold fast to the belief, stemming from a dualist ideology, that at least the students who are succeeding in applying procedures correctly understand what they're doing-surely they must understand something of what they're doing!-but in our heart of hearts, we know we're not getting through.

The Lexical Support System

In the opening paragraph of this chapter I posed a riddle: "how ... could students' persistent errors possibly indicate anything other than that algebra is difficult to understand?" The answer is that there is nothing in the record of students' past performance-their errors or their correct work-that gives us any idea whatsoever as to whether algebra is difficult to understand. That's because students, by and large, haven't been engaged in the project of creating meaning in their development of algebraic skills. Rather, they've been refining their sensitivity to patterns in the visual display. So we don't know how they would fare with a curriculum designed to support meaning making in the manipulation of algebraic symbols. In this section, we finally get down to outlining how such an instructional approach can be structured.

The curricular challenge for making algebra structurally meaningful is to create discursive practices that intervene between students' visual apprehension of algebraic notation and their manipulation of expressions and equations, establishing a new layer of explicit justification for the latter. As the centerpiece for such a discourse I sketch out a Lexical Support System (LSS) (Kirshner, 1998; Kirshner & Awtry, 2004) that makes explicit the structural aspects of expressions and equations that are only tacitly assumed in a traditional approach. This curricular approach is currently in the process of refinement and testing, and is proposed for wide-scale implementation in secondary school algebra instruction. Rather than go through the curriculum in systematic detail, I provide an overview of key points, and sketch out instructional phases with a view to giving the "flavor" of the approach.

The LSS approach transforms the simplification of expressions and the solving of equations by providing a structural vocabulary that enables more rigorous description of algebraic rules and procedures. The purpose is to provide for step-by-step justification of algebraic procedures so that explicit reasoning can come to imbue the student's relationship to the symbol system. The LSS curriculum is serial and sequential-a mastery curriculum, in which a topic is learned thoroughly and completely before progressing to the next. In this respect it is unlike the traditional spiral curriculum which revisits prior topics repeatedly, each time at a higher level of sophistication (hopefully).

Algebra has two faces. The *empirical face* points outward toward domains of reference, toward modeling phenomena in the world, toward application, toward number, quantity, and shape. The *structural face* points inward to the logical infrastructure, to the grammar of rules and procedures abstracted from external realms of interpretation. The LSS is a structural curriculum. It deals only with the formal alphanumeric symbol system, disregarding graphs, tables, natural language description, physical models, geometric analogies, and so on. So the LSS is not the "whole enchilada" of algebra instruction. Rather, it is proposed as the inner core of a much broader and more extensive curriculum that includes both faces of algebra. Still, it is sensible for us to focus curricular attention on this face independently, to ensure that algebraic structure is properly represented for our students. For, as Bell (1936) described the structural mandate:

> The very point of elementary algebra is simply that it *is* abstract, that is, devoid of any meaning beyond the formal consequences of the postulates laid down for the marks. ... Algebra stands upon its own feet as a "hypothetico-deductive system." (p. 144)

The LSS approach is comprised of 4 phases. The foundation, introduced in **Phase I**, is an explicit accounting of order of operations conventions. Such an account specifies *aggregation markers* that visually "force" a parse of the expression, and a hierarchy of operations that applies in the absence of aggregation markers. Aggregation markers include the expected brackets, braces, and parentheses, but also superscription (compare $3x^{2+y}$ and $3x^2+y$) and the vinculum (horizontal line) used in fractions and radicals (compare $\sqrt{x}+5$ and $\sqrt{x+5}$).

The operation hierarchy is neatly summarized in terms of *operation levels* ascending from 1 to 3:

Level 1 addition and subtraction

Level 2 multiplication and division

Level 3 exponentiation and radical

where (a) Higher level operations are precedent (e.g., $1+3x^2=1+[3(x^2)]$)

and (b) If adjacent operations are of equal level, the operation on the left is precedent (e.g., $5-3+1=(5\cdot 3)+1$)

Order of operations is woefully neglected in traditional approaches that often rely on the PEMDAS acronym (Parentheses, Exponentiation, Multiplication, Division, Addition, Subtraction) that mixes together aggregation markers and the hierarchy of levels, providing an incomplete and inexplicit account of each. Such practices persist in the traditional curriculum because, relying on visual cues, students are generally able to parse expressions correctly (Kirshner, 1989). However, explicit knowledge of order of operations turns out to be a crucial key into the realm of explicit reasoning about transformational rules and procedures in algebra.

Phase 1 begins with presentation of order of operations rules which are practiced through expression evaluation exercises that require students to justify their step-by-step calculation of binary combinations in terms of precedence rules. Upon the foundation of order of operations is erected the basic lexical elements of the LSS. The *principal operation* of an expression is defined as the least precedent operation according to the order of operation rules. For instance, the principal operation of $3x^{2+y}$ is multiplication because that operation has least precedence within the given expression-it would be the last operation performed if we were evaluating the expression for given values of x and y. *Principal subexpressions* are then defined as the parts of an expression joined by the principal operation (so 3 and x^{2+y} are the principal subexpressions of $3x^{2+y}$). Recursively, each subexpression can itself be parsed into principal subexpressions yielding a complete structural description of an expression. In this phase, students use the language of "principal subexpressions," "next-most principal subexpressions" etc. to talk through the structure of operations from most principal to most precedent. Because this is a mastery approach, we work Phase 1 through to expressions of arbitrary complexity. For instance, in a small-scale study designed to organize and initially try out this approach, students were asked (and able!) to describe the structure of complex expressions like

$$\sqrt{\frac{13^2-5\times\sqrt{250-15^2}}{\dfrac{23-4^2}{2^2+3}}}$$

The language of principal operations and principal subexpressions allows us to formally define some of the basic terminology that we use in algebra classes, but without structural grounding: *Terms* (*factors*) are the principal subexpressions of an expression whose principal operation is addition (multiplication).

Phase II introduces *transformational rules* of algebra (e.g., associative, distributive, difference of squares, etc.). Transformational rules are approached as templates that allow one to operate on an expression of a specified syntactic form to produce an expression of another syntactic form. For instance, $(xy)^2=x^2y^2$ takes an expression whose principal operation is exponentiation and whose base is a product and transforms it into an expression whose principal operation is multiplication, and whose principal subexpressions (i.e., factors) both have exponentiation as their principal operations. This enables transformational rules to be rigorously applied. For instance, we can apply the $(xy)^2=x^2y^2$ rule to $[(a+b^2)c^2]^{3+m}$ not (only) because of its visual relationship to the left-hand side of the rule, but because $(xy)^2$ forms an explicit structural template that $[(a+b^2)c^2]^{3+m}$ fits. This explicit verbal accounting of transformational rules intervenes in the spontaneous visual association between right-hand and left-hand side that seems to arise for visually salient rules, demanding that the two sides be individually analyzed. Thus mal-rules, which are visual overgeneralizations of visually salient rules, should be reduced or completely eliminated through this approach.

Phase III extends the LSS approach to the structural description of standard transformational tasks. For example, we define *to factor* as "to transform an expression whose principal operation is not multiplication to an expression whose principal operation is multiplication." Phase III instruction requires students to reason about algebraic tasks based solely on such formal task descriptions. For instance, with only the definition of "to factor" at their disposal, students select out *transformational rules that* could be used to factor an expression (e.g., $xy \pm xz = x(y \pm z)$, $x^2 - y^2 = (x-y) \cdot (x+y)$, and $(xy)^2 = x^2y^2$, etc.). This skill draws on the explicit description of transformational rules developed in Phase II. Phase III also includes multi-step applications like simplifying fractional expressions that may require extensive preparation before the final simplification can be performed. For instance, if *simplifying a fractional expression* is defined as "cancelling a common factor from the numerator and denominator of the fraction," then factoring numerator and denominator are logical precursors to the final simplifying step. Davis (1984) pointed out that students' extended derivations in elementary algebra usually are constituted as "Visually Moderated Sequences" which "can be thought of as a visual cue V_1 which elicits a procedure P_1 whose execution produces a new visual cue V_2, which elicits a procedure P_2,...and so on" (p. 35). The LSS curriculum overlays a *structural discourse that* enables solution strategies to be mapped out and discussed in advance, and deliberately implemented.

Phase IV extends the structural approach to the solving of equations. Solving equations involves applying two kinds of rules. Much of the work with equations is simplifying the individual expressions given on the right-hand or on the left-hand side of the equal sign. This is exactly what already has been addressed in the first 3 phases. Additionally, the law of equations enables one to do the same thing to both sides of an equation (with appropriate restrictions).

From a structural perspective there are several different cases, each with its own structural dynamics of solution: single occurrence of a variable; multiple occurrences of a single variable, all to the same degree; more than a single degree of a variable; and systems of equations. Each case can be dealt with in a comprehensive fashion before moving on to the next. For instance, equations with a single occurrence of an unknown all can be solved by applying a strategy of "undoing" each operation using inverse operations starting with the principal operation, and working backward through the expression to the most precedent operation. In the mastery curriculum mode, one works such a strategy through to solving complex equations like $\sqrt{\frac{2x^3+21}{3}} = 5$ before moving on to the other cases.

We do not stop at simple linear functions, as is typical of traditional curricula.

This completes the overview of instructional phases. In our initial testing of this approach, and in my own prior informal experimentation, it seems the structural perspective is not difficult to master even for weak students, given sustained attention and practice. The following contrived episode, similar to many I've engaged in when using the LSS approach, illustrates the sort of communicational possibilities opened up by these more rigorous discursive practices. This interaction involves a student's erroneous cancellation of the 3s in

$$\frac{3x^2+1}{3y-2} = \frac{x^2+1}{y-2}$$

Teacher: What rule are you using in this step?

Student: The cancellation rule for fractions.

Teacher: Can you remind me what that rule is?

Student: It's the rule that allows canceling a common factor of the numerator and denominator of a fractional expression.

Teacher: Okay, let's take a look at it. What have you canceled?

Student: The threes, because they're factors, they're multiplied.

Teacher: Good, they are indeed factors, but are they factors of the *numerator and denominator*? Let's check. What is the principal operation of the numerator?

Student: Let's see, there's an exponentiation, a multiplication, and an addition. So the principal operation is addition, the least precedent one according to the hierarchy of operations.

Teacher: Good, now what are the principal subexpressions called in this case?

Student: They're called terms. ...Oh, I see, it has to be a factor of the *whole* numerator and denominator to be canceled; not just part of it.

Such communicative possibilities can be contrasted with traditional algebra instruction in which students and teachers talk past each other as they use words like "term" and "factor" without structural grounding. Perhaps the teacher admonishes the student to make sure they are canceling factors. But the structural distinction, so clear and tangible for the teacher, is not conveyed to the student. Instead, the student learns only that they have done something wrong and need to do something different. Absent an understanding of the structural fundamentals, what gets recorded for the student is something about the visual shape of incorrect and correct applications. Eventually, with persistence, the visual patterns become sufficiently refined as to constrain incorrect applications. In this way, what begins as an opportunity for communication of structural information is reduced to support for mindless matching of visual patterns.

Enculturating Students to Algebra as a Formal Discourse

The title of this paper suggests that structural understanding of algebra doesn't need to be as difficult for out students as it has seemed to be. I've tried to support this position by showing that students' persistent errors reflect disengagement from structural meaning, not difficulty in understanding structural algebra. In this respect, the problem of algebra is not so much a conceptual one, as a cultural one. Because we've misconstrued students' correct work as indicating understanding, we've dropped the ball in presenting a structurally sound and formally rigorous version of algebra to our students. The LSS approach involves attending more carefully to the discursive structure of our curriculum and to the culture of students' participation.

Elsewhere (Kirshner, 2002; Kirshner & Awtry, 2004) I've argued that learning a culture has taken a back seat to learning concepts, with goals for the former often left confused and undifferentiated. For that reason, I want to be especially careful to delimit the precise cultural goals addressed by the LSS curriculum. For there are many facets to structural algebra, and a serious and sustained instructional effort is needed to ensure that they are addressed in a coherent and progressive fashion.

Ernest (1998) identified foundational aspects of mathematical method-logicality and formality-that enable us to assert the validity of mathematical claims. Logicality relies on explicit processes of inferential reasoning; formality, on rigorous application of uninterpreted rules. Both of these aspects of mathematical method constitute important goals for mathematics education. We want our students to be able to reason logically, and we want our students to be able to exploit the power of symbols to preserve truth through syntactically defined transformations.

The last concerted effort to orient school algebra around structural goals was in the New Math era of the 1960s and 1970s. The New Math was oriented by "the concepts of set, relation, and function and by judicious use of broadly applicable mathematical processes like deductive reasoning and the search for patterns" (Fey & Graeber, 2003, p. 524). However, the New Math is widely regarded to have been "excessively formal, deductively structured, and theoretical.... fail[ing] to meet the needs for basic mathematical literacy of average and low ability students" (NACOME, 1975, p. ix).

In analyzing the difficulties encountered with the New Math, it is worth noting the logicist agenda it reflected (Ernest, 1985). Indeed, an explicit intention of the New Math was to distribute part of the emphasis on deductive reasoning from the geometry curriculum to algebra:

> One way to foster an emphasis upon understanding and meaning in the teaching of algebra is through the introduction of instruction in deductive reasoning. The Commission [on Mathematics] is firmly of the opinion that deductive reasoning should be taught in all courses in school mathematics and not in geometry alone. (College Entrance Examination Board, 1959, p. 23)

But deductive reasoning is notoriously difficult for adults, let alone adolescents (Evans, 1982). In particular, conditional reasoning-deductions based on *if* p *then* q-are very confusing. When given *if* p *then* q, the logical principle Modus Ponens (assert p and deduce q) is straightforward; however, its contrapositive Modus Tolens (assert *not* q, deduce *not* p), though valid, is unobvious. And invalid variations (assert *not* p, deduce *not* q) and (assert q, deduce p) are accepted as valid by a plurality of adults (Evans, 1982). The focus on inferential logic likely explains why the New Math seemed to be successful only for talented, college-bound students.

In contrast, the LSS focuses on formal methods of mathematics-structural analyses of expressions and equations, and rigorous application of transformational rules. Now, there is a sense in which an algebraic derivation can be considered a logical proof of equivalence. For instance, the derivation $3x^2-27=3x^2-3\cdot 9=3(x^2-9)=3(x^2-3^2)=3(x-3)(x+3)$ proves the equivalence of $3x^2-27$ and $3(x-3)(x+3)$. But in its logical structure, such derivations rely on biconditional reasoning rather than conditional reasoning: $3x^2-27$ is true if and only if $3(x-3)(x+3)$ is true; and each step of the derivation is logically reversible. Biconditional reasoning is easy for adults and for children, as the truth or falsity of either term validly implies the truth or falsity, respectively, of the other. All four of the inferential possibilities noted above are valid. So the LSS curriculum sets out to address limited but achievable structural goals. It is an introduction to structural algebra appropriate for the secondary school, one that may provide a foundation for many students to enter further into the realm of mathematical culture as they progress in their school experience.

Given the perceived failure of the New Math curriculum, and the subsequent record of mindless symbol manipulation in elementary algebra, it is not surprising that contemporary mathematics educators are shy about pursuing structural goals-even, denying the possibility that formal work in algebra can be meaningful for students:

> acts of generalization and gradual formalization of the constructed generality must precede work with formalisms-otherwise the formalisms have no source in student experience. The current wholesale failure of

> school algebra has shown the inadequacy of attempts to tie the formalisms to students' experience after they have been introduced. It seems that, "once meaningless, always meaningless." (Kaput, 1995, pp. 74-75)

NCTM's (2000) *Principles and Standards for School Mathematics* echos these sentiments, thereby authorizing empirical algebra as the sole agenda for algebra instruction:

> In general, if students engage extensively in symbolic manipulation before they develop a solid conceptual foundation for their work, they will be unable to do more than mechanical manipulation (NRC 1998). The foundation for meaningful work with symbolic notation should be laid over a long time. (p. 39)

By reanalyzing this history, we can see new possibilities for a successful structural curriculum to complement the burgeoning interest in empirical algebra spawned by new technologies that enable us to hot-link symbolic, graphical, and tabular representations to real-world data sources. Empirical algebra, by itself, is not enough. The bird of algebra needs both of its wings to soar.

References

Bell, E. T. (1936). The meaning of mathematics. In W. D. Reeve (Ed.), *The place of mathematics in modern education* (pp.136-181). NCTM Eleventh Yearbook, New York: Bureau of Publications, Teachers College, Columbia University.

Bereiter, C. (1991). Implications of connectionism for thinking about rules. *Educational Researcher*, **20**, pp.10-16.

Bereiter, C., & Scardamalia, M. (1996). Rethinking learning. In D. Olson &. N. Torrance (Eds.), *Handbook of education and human development: New models of learning, teaching and schooling* (pp.485-513). Cambridge, MA: Basil Blackwell.

Blume, G. W., & Heckman, D. S. (1997). What do students know about algebra and functions. In P. A. Kenney & E. A. Silver (Eds.), *Results from the sixth mathematics assessment of the National Assessment of Educational Progress* (pp.225-277). Reston, VA: NCTM.

Brooks, R. A. (1991). Intelligence without representation. *Artificial Intelligence,* **47**, pp.139-159.

Carry, L. R., Lewis, C., & Bernard, J. E. (1980). *Psychology of equation solving: An information processing study* (Tech Rep. SED 78-22293). Austin: The University of Texas at Austin, Department of Curriculum & Instruction.

Clancey, W. J. (1999). *Conceptual coordination: How the mind orders experience in time*. Mahwah, N.J.: L. Erlbaum Associates.

College Entrance Examination Board. (1959). *Report of the Commission on Mathematics*. New York: CEEB.

Davis, R. B. (1984). *Learning Mathematics*. Norwood, NJ: Ablex Publishing Corporation.

Dreyfus, H. L. (2002). Intelligence without representation-Merleau-Ponty's critique of mental representation: The relevance of phenomenology to scientific explanation. *Phenomenology and the Cognitive Sciences*, **1(4)**: pp.367-383.

Dupuy, J-P. (2000). *The mechanization of the mind: On the origins of cognitive science*. Translated by M. B. DeBevoise. Princeton, NJ: New French Thought.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *The Journal of Mathematical Behavior*, **1(2)**, pp.7-26.

Ernest, P. (1985). The philosophy of mathematics and mathematics education. *International Journal of Mathematics, Education, Science, and Technology*, **16(5)**, pp.603-612.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany, NY: State University of New York Press.

Estep, M. (2003). *A theory of immediate awareness: Self-organization and adaptation in natural intelligence*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Evans, J. St. B. T. (1982). *The Psychology of Deductive Reasoning*. London: Routledge & Kegan Paul.

Fey, J. T., & Graeber, A. (2003) From the New Math to the Agenda for Action. In G. Stanic & J. Kilpatrick (Eds.) *A history of school mathematics* (Vol. 1, pp.521-558). Reston, VA: National Council of Teachers of Mathematics.

Gee, J. P. (1992). *The social mind: Language, ideology, and social practice*. New York: Bergin & Garvey.

Haberlandt, K. (1997). *Cognitive psychology*. Boston: Allyn & Bacon.

Haugland, J. (1985). Semantic engines: An introduction to mind design. In J. Haugland (Ed.), *Mind design* (pp.1-34). Cambridge, MA: : The MIT Press.

Kaput, J. J. (1995). A research base supporting long term algebra reform. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, **vol 2** (pp.71-108).

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education*, **20**(3), pp.274-287.

Kirshner, D. (1998) *Discursive algebra: Explicit talk about implicit processes*. Paper and poster presented to *Visions of algebra in diverse instruction*, a working group of the Twentieth Annual Meeting of the International Group for the Psychology of Mathematics Education, North American Chapter. Raleigh, NC. October 31.

Kirshner, D. (2001). The structural algebra option revisited. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp.83-98). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kirshner, D. (2002). Untangling teachers" diverse aspirations for student learning: A crossdisciplinary strategy for relating psychological theory to pedagogical practice. *Journal for Research in Mathematics Education,* **33**(1), pp.46-58.

Lloyd, D. (1989). *Simple minds: Mental representations from the ground up*. Cambridge, MA: MIT Press.

Matz, M. (1980). Towards a computational theory of algebraic competence. *The Journal of Mathematical Behavior*, **3**, pp.93-166.

National Advisory Committee On Mathematics Education. (1975). *Overview and analysis of school mathematics grades K-12*. Washington, D.C.: Conference Board of the Mathematical Sciences.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (1998). *High school mathematics at work: Essays and examples of for the education of all students*. Washington, DC: National Academy Press.

Rumelhart, D. E., Hinton, G. E., & Williams, R. J. (1986). Learning internal representations by error propagation. In D. E. Rumelhart, & J. L., McClelland (Eds.), *Parallel distributed processing: Explorations in the microstructure of cognition* (pp.318-362). Cambridge, MA: MIT Press.

Sleeman, D. (1986). Introductory algebra: A case study of student misconceptions. *The Journal of Mathematical Behavior*, **5**(1), pp.25-52.

Thompson, P. W. (1989). Artificial intelligence, advanced technology, and learning and teaching algebra. In C. Kieran & S. Wagner (Eds.), *Research agenda for mathematics education: Research issues in the learning and teaching of algebra* (pp.135-161). Reston, VA: National Council of Teachers of Mathematics. Hillsdale. NJ: Lawrence Erlbaum Publishers.