A fourth-order accurate alternating group explicit parallel algorithm for the convection-diffusion equation

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ABSTRACT

It is well known that many transport problems appearing in the study of fluid motion, heat transfer, astrophysics, oceanography, meteorology, semiconductors, hydraulics, pollutant & sediment transport and chemical engineering, can be reduced to convection-diffusion equation subject to certain initial and boundary conditions. Several effective high order accutate schemes have been proposed to simulate the equation, such as ADI, Crank-Nicolson, operator splitting method, Eulerian-Lagrangian method, Finite Element method, Boundary Element method, TVD method, high order upwind biasing method, High Order Compact schemes and so on. However, many of these schemes need to cost more computing time and data storage when applied to model the large scale transport problems, since they are implicit or more complicated and expensive. With respect to the time integration, there are two mainly numerical methods for Computational Fluid Dynamics(CFD) to solve the convection-diffusion equations, implicit and explicit schemes. Implicit schemes often exhibit unconditional stability for governing equations, but involve more complex code, more computational time and storage per iteration. Explicit schemes are usually conditionally stable but are relatively easy to program, require less computational time and storage per iteration. In CFD, the numerical schemes which possess of the advantages of both explicit and implicit algorithm are intriguing. Hence, a fourth-order alternating group explicit (AGE) parallel scheme for solving the unsteady convection-diffusion equation is presented and discussed in this paper. It was derived basing on the local series expansion method which is an effective approach for derivation of general formulas to approximate the partial differential equation. The approach is briefly outlined as below:

For the sake of simplicity, we consider the one-dimensional convection-diffusion equation:

$$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x}=D\frac{\partial^2 \phi}{\partial x^2}, \qquad 0\le x\le 1 \ , \ 0\le t\le T \tag{1}$$

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The solution domain $[0,1]\times[0,T]$ of the problem is covered by a mesh of grid-lines $x_i = i\Delta x$ and $t_k = k\Delta t$ parallel to the space and time coordinate axes, respectively. Approximations ϕ_i^k to $\phi(i\Delta x, k\Delta t)$ are calculated at point of intersection of these lines, $(i\Delta x, k\Delta t)$ which is referred to as the (i,k) grid-point. In the spatial and temporal subdomain $[x_{i-2}, x_{i+2}]\times[k, k+1]$ shown in Fig.1, the value of ϕ_i^{k+1} can be approximated by the (N)th-oder taylor series expansion as:

$$\phi(x,\tau) = \sum_{j=0}^{N} a_j(\tau)\cdot \chi^j \tag{2}$$

where $\chi = x - x_i$, $\tau = t - t_k$, $a(\cdot)$ is the unknown function of time and N is the order of the taylor series expansion. In present paper, $N = 4$ is chosen for deriving the AGE formulas composed of six points in two temporal levels for the convection-diffusion equation:

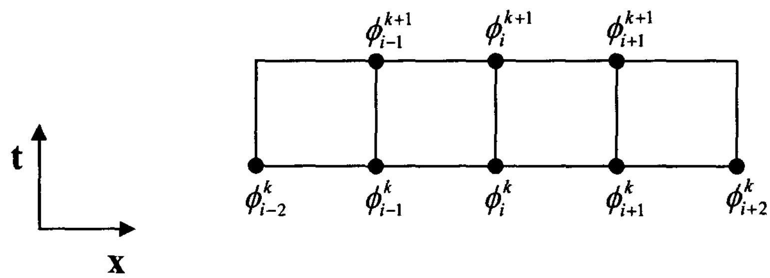


Fig.1 The computational schematics of the local series expansion method

Substituting Eq.(2) into Eq.(1), and let $\chi = 0$, $\tau = \Delta t$, we can obtain

$$\begin{aligned}\phi_i^{k+1} &= a_0(0) - u\cdot\Delta t\cdot a_1(0) + (2\cdot D\cdot\Delta t + u^2\cdot(\Delta t)^2)\cdot a_2(0) + \left[-6\cdot D\cdot u\cdot(\Delta t)^2 - u^3\cdot(\Delta t)^3\right]\cdot a_3(0) \\ &\quad + \left[12\cdot D^2\cdot(\Delta t)^2 + 12\cdot D\cdot u^2\cdot(\Delta t)^3 + u^4\cdot(\Delta t)^4\right]\cdot a_4(0)\end{aligned} \tag{3}$$

If $N = 4$, It is shown that there are five unknown constants $a_0(0)$, $a_1(0)$, $a_2(0)$, $a_3(0)$ and $a_4(0)$ in Eq.(3). If five different nodal values of ϕ are chosen to be substituted into Eq.(2), the values of $a_0(0)$, $a_1(0)$, $a_2(0)$, $a_3(0)$ and $a_4(0)$ can be determined and the specific expression of Eq. (3) can be obtained.

When choosing the different five nodal values of ϕ shown in Fig.1, we can obtain three formula:

$$A_{i-1}^{k+1}\cdot\phi_{i-1}^{k+1} + A_i^{k+1}\cdot\phi_i^{k+1} + A_{i+1}^{k+1}\cdot\phi_{i+1}^{k+1} = A_{i-2}^{k}\cdot\phi_{i-2}^{k} + A_{i-1}^{k}\cdot\phi_{i-1}^{k} + A_i^{k}\cdot\phi_i^{k} \tag{4}$$

$$B_{i-1}^{k+1}\cdot\phi_{i-1}^{k+1}+B_{i}^{k+1}\cdot\phi_{i}^{k+1}+B_{i+1}^{k+1}\cdot\phi_{i+1}^{k+1}=B_{i-1}^{k}\cdot\phi_{i-1}^{k}+B_{i}^{k}\cdot\phi_{i}^{k}+B_{i+1}^{k}\cdot\phi_{i+1}^{k} \tag{5}$$

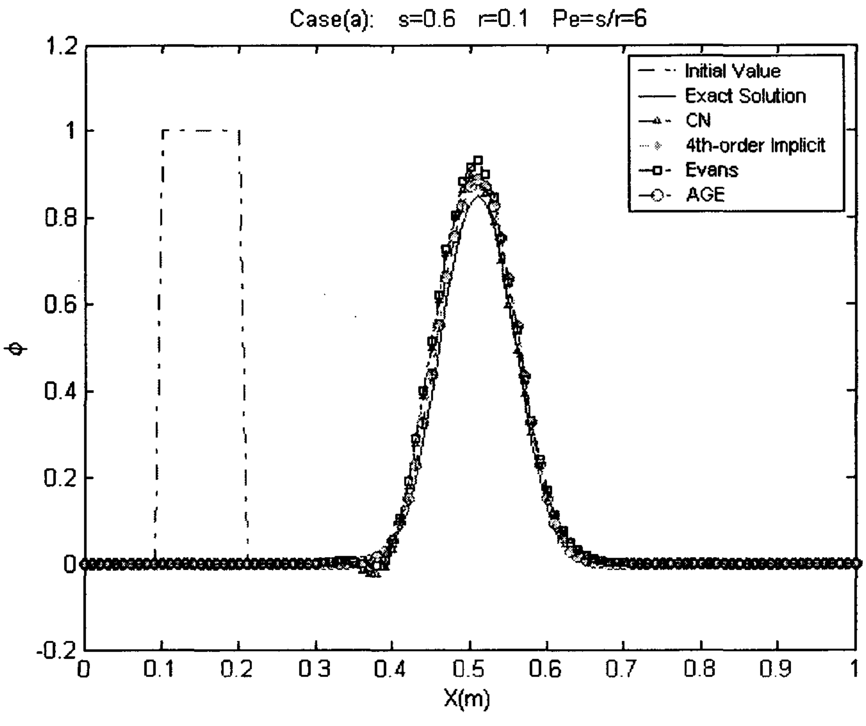
$$C_{i-1}^{k+1}\cdot\phi_{i-1}^{k+1}+C_{i}^{k+1}\cdot\phi_{i}^{k+1}+C_{i+1}^{k+1}\cdot\phi_{i+1}^{k+1}=C_{i}^{k}\cdot\phi_{i}^{k}+C_{i+1}^{k}\cdot\phi_{i+1}^{k}+C_{i+2}^{k}\cdot\phi_{i+2}^{k} \tag{6}$$

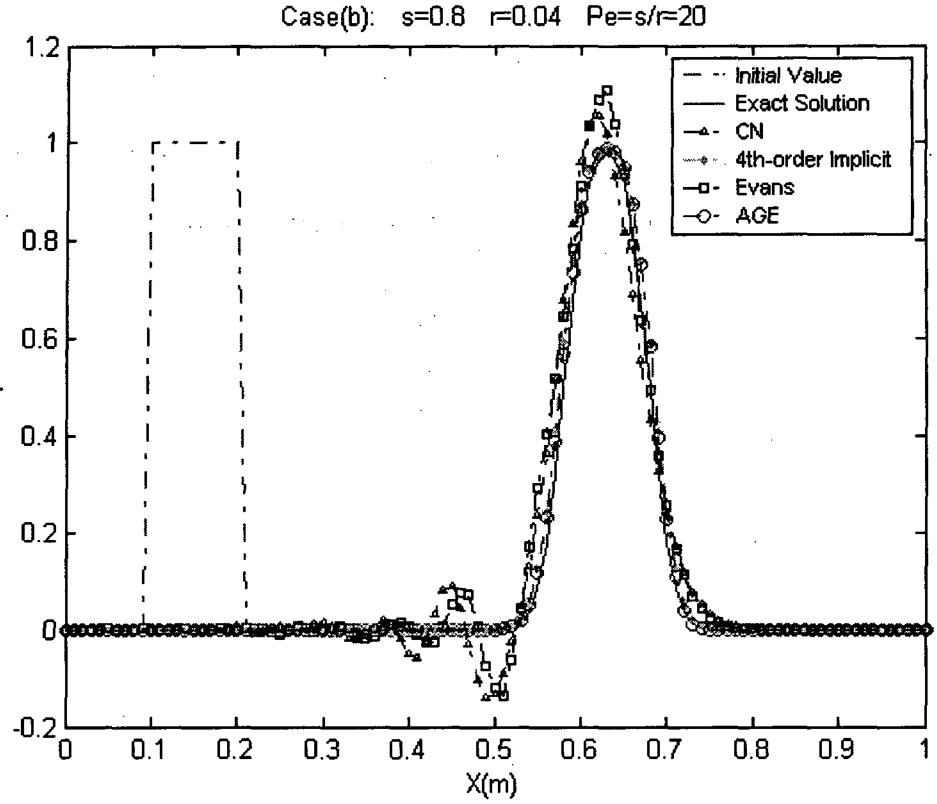
where A,B,C are coefficients. It is shown that the Eq.(4), Eq.(5) and Eq.(6) are are implicit formulas. Combining the use of the Eq.(4), Eq.(5) and Eq.(6), we can obtain the explicit version of the system (4)-(6):

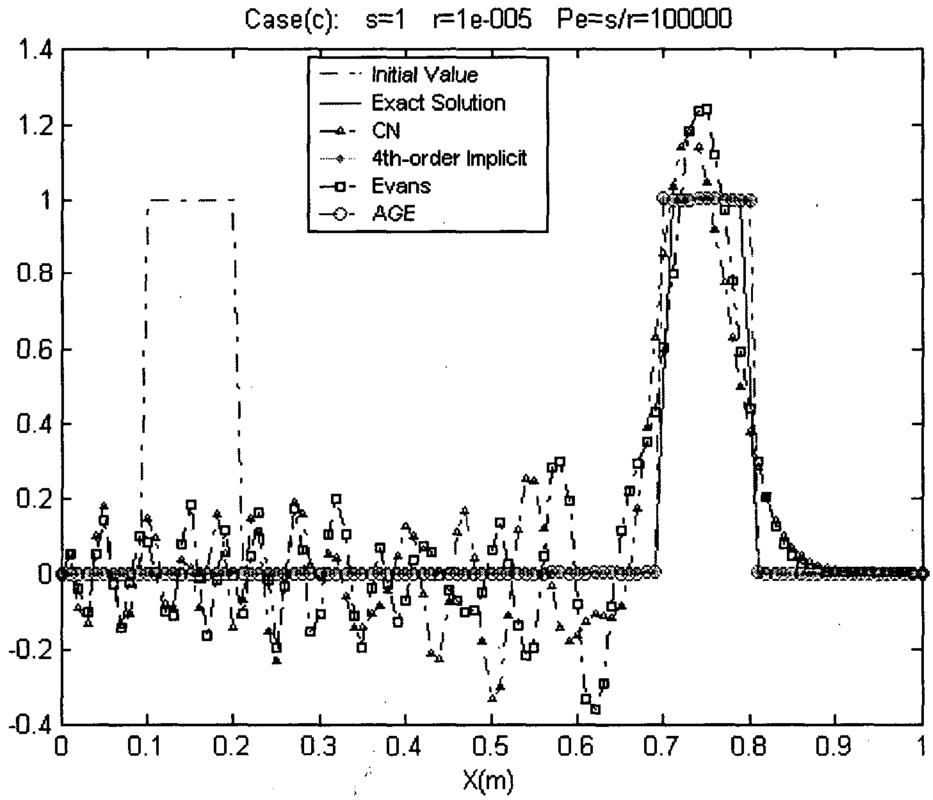
$$\begin{bmatrix}\phi_{i-1}^{k+1}\\ \phi_{i}^{k+1}\\ \phi_{i+1}^{k+1}\end{bmatrix}=E\begin{bmatrix}\phi_{i-2}^{k}\\ \phi_{i-1}^{k}\\ \phi_{i}^{k}\\ \phi_{i+1}^{k}\\ \phi_{i+2}^{k}\end{bmatrix} \tag{7}$$

Where E is the matrix coefficients derived in the full paper and Eq.(7) is the proposed fourth-order alternating group explicit (AGE) parallel scheme for solving the unsteady convection-diffusion equation

The validity of the proposed method is tested by a one-dimensional convection-diffusion equations with a Sharpe pulse type concentration and also compared to the conventional schemes include: (i) the Lax – Wendroff (LW) scheme; (ii) the Crank-Nicolson (CN) scheme, (iii) the fourth-order implicit scheme (4th Implicit) and (iv)the D.J.Evans scheme (Evans) described in D.J.Evans and A.R.B. Abdullah [2]. Three cases (for different $s=\frac{u\Delta t}{\Delta x}$ and $r=\frac{D\Delta t}{\Delta x^2}$) were considered in this paper which described the convection dominated flow problems and the numerical results are shown below.







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