Optimal System Design of Consecutive-k-out-of-n:F System

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The linear consecutive-k-out-of-n:F system consists of n components ordered linearly and fails if and only if at least k consecutive components fail. We assume that the components are independent and identical exponentially distributed. This paper develops a model to calculate the expected cost per unit time of a linear consecutive-k-out-of-n:F system. The optimization problem we consider regards the choice of the system structure parameter k to minimize the expected cost per unit time.

1. Introduction

Consecutive-k-out-of-n:F system models have been proposed for the system reliability evaluation and for the design of microwave relay stations in telecommunications, oil pipeline systems, integrated circuits, vacuum systems in accelerators, computer-ring networks, and space craft relay stations. A linear consecutive-k-out-of-n:F system consists of n components ordered linearly. The system fails if and only if at least k consecutive components fail.

Since the consecutive-k-out-of-n:F system was first introduced by Kontoleon [8], the system has been extensively studied by reliability engineers and researchers. Studies on this system were reviewed by Chao, Fu and Koutras [4], Chang, Cui and Hwang [1] and Kuo and Zuo [9].

One goal of reliability engineers is to increase the system reliability. If the system reliability predicted does not satisfy the reliability engineers, they have to find a reasonable method to increase it. A method has two stages: system design optimizations and system operation optimizations.

System design optimizations are executed in the system design phase and include an optimal assignment problem and an economic system design problem. The optimal assignment problem includes a reliability assignment to maximize the system reliability through optimally assigning components to positions in the system. Derman, Lieberman and Ross [5], Du and Hwang [6], and Hwang and Pai [7] provided the invariant optimal design (permutation) which maximizes the system reliability. Zuo and Kuo [15] summarized the results available for the invariant optimal designs of the consecutive-k-out-of-n:F system. Another optimal design problem, economic system design, economically deals with system parameters such as p_i, k, and n. Chao and Lin [3] introduced the system structure design aspect into the problem for the consecutive-k-out-of-n:F system. Chang and Hwang [2] found the best structure, a set $\{p_1, \ldots, p_n\}$, and its assignment that maximizes the system reliability under a fixed budget. Yun and Kim [12] studied the system structure parameter k to minimize the expected cost per unit time for a consecutive-k-out-of-n:F system with $(k-1)$

-step Markov dependence.

Second optimizations are executed in the system operation phase in which optimal usage (optimal repair/replacement policy) is determined. The maintenance can increase not only system reliability but also enormous expense. Recently, Yun, Kim and Yamamoto [14] and Yun, Kim and Jeong [13] introduced a maintenance design to the consecutive-k-out-of-n:F system so as to find an economic way to increase system reliability through maintenance.

We concentrate on a economic design problem for the linear consecutive-k-out-of-n:F system with i.i.d. exponentially distributed components. In that case, we determine the optimal system performance k to minimize the expected cost per unit time.

This paper is arranged as follows. Section 2 gives an cost model for the expected cost per unit time. Section 3 describes computational experiments which investigate the relationship between the some parameters and the optimal k. In section 4 a brief summary of conclusions is provided.

2. Cost Model

The optimization problem we are going to tackle regards the choice of the system performance, k. The objective is to minimize the expected cost per unit time.

If there are failed components at system failure, we must need $(C_0+iC_R(k))$ as the total cost, where $C_R(k)$ is maybe a increasing function for k as we may have more replacement cost for the pump with a large capacity.

The following assumptions and notation are used in this paper.

Assumptions

1. The system consists of n identical and independent components which lifetimes have exponential distribution.
2. The failed components are replaced with new ones when the system fails.
3. Replacement cost per component $C_R(k)$ is a increasing function for k, for example $C_R(k)=C_1+C_2k$.
4. Failures occur one at a time.
5. Replacement time is negligible.

Notation

n	number of components in the system
k	minimum number of consecutive failed components for system failure
λ	failure rate of a component
T_k	lifetime of the system
N_k	number of failed components at system failure
C_0	cost for system breakdown
$C_R(k)$	replacement cost per component
$S_d(k)$	expected length of a cycle
$S_c(k)$	expected total cost in a cycle
$C(k)$	expected cost per unit time

$dC(k)\equiv C(k+1)-C(k)$

$\lfloor x \rfloor$	greatest integer lower bound of x

We assume that all failed components are replaced with new ones at system failure so that we can see system failure as a renewal point. Therefore, from the well-known renewal theory, the expected cost per unit time is given by

$$CR(k)=\frac{S_c(k)}{S_d(k)}=\frac{C_0+E(N_k)C^{(k)}}{E(T_k)} \quad (1)$$

From the closed expression of the reliability of the system in Lambiris and Papastavridis [10], we have the following expression of the expected lifetime of the system of a linear consecutive-k-out-of-n:F system with i.i.d. components.:

$$E(T_k) = \int_0^{\infty} R(t)dt$$

$$= \sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \int_0^{\infty} e^{-(n-i+j)\lambda t} dt$$

$$= \frac{1}{\lambda}\left[\sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1}{n-i+j}\right] \quad (2)$$

where

$$M = \begin{cases} n - \frac{n-1}{k} + 1 & \text{if } n+1 \text{ is a multiple of } k \\ n - \left\lfloor \frac{n+1}{k} \right\rfloor & \text{otherwise} \end{cases}$$

and

$$A(i) = \sum_{s=0}^{\lfloor i/k \rfloor} \binom{n-i+1}{s} \binom{n-sk}{n-i} (-1)^s.$$

Papastavridis [11] identified the probability mass function of the number of failed components in a linear consecutive-k-out-n:F system. We can express the expected number of failed components at system failure as follows:

$$E(N_k) = \sum_{m=k}^{n} m \cdot Pr(N_k = m)$$

$$= \sum_{m=k}^{n} \sum_{x=k}^{m^*} \sum_{i=1}^{n-x+1} \sum_{j=0}^{i^*} \Big[2k - x \Big/ \binom{n}{m} \Big) \cdot B(j, i-1-j)$$

$$\cdot B(m-x-j, n+j-i-m+1) \Big] \quad (3)$$

where, $m^* = min\{m, 2k-1\}$ and $i^* = max\{0, i-2\}$ and

$$B(i,j) = \sum_{s=0}^{j} \binom{j}{s} \binom{i+j-1-sk}{i-sk} (-1)^s.$$

From Equations (1), (2), and (3), the expected cost per unit time can be expressed by the form:

$$C(k) = \lambda \Big[C_0 + C^{(k)} \sum_{m=k}^{n} \sum_{x=k}^{m^*} \sum_{i=1}^{n-x+1} 2k - x \Big/ \binom{n}{m}$$

$$B(j, i-1-j) B(m-x-j, n+j-i-m+1) \Big]$$

$$\Big/ \left[\sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1}{n-i+j} \right]. \quad (4)$$

The objective is to minimize the expected cost per unit time. We can easily find the optimal system performance parameter k^* as verifying from 1 to system size n, using a statistical package, for example MATHEMATICA.

It is difficult to analytically get the property of optimal k because of the complexity of the cost function. The numerical search using by MATHEMATICA can figure out the cost function. Figure 1 shows how the expected cost per unit time varies with k. Figure 2 shows differences of the cost function $dC(k) \equiv C(k+1) - C(k)$ and illustrates that the cost function has triple infection point. These point out the expected cost per unit time can be *multi-modal.*

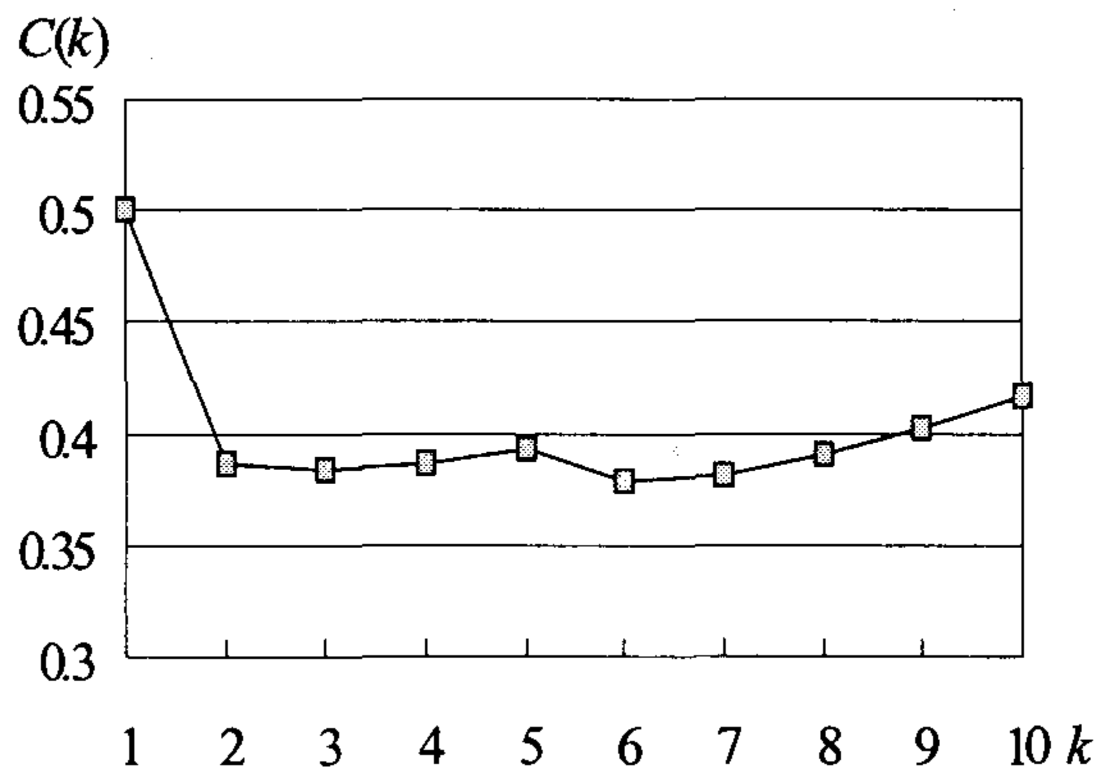


Figure 1. $C(k)$ vs k (n=10, $\lambda = 0.02$, $C_0 = 5$, and $C_R(k) = 1 + 0.5k$)

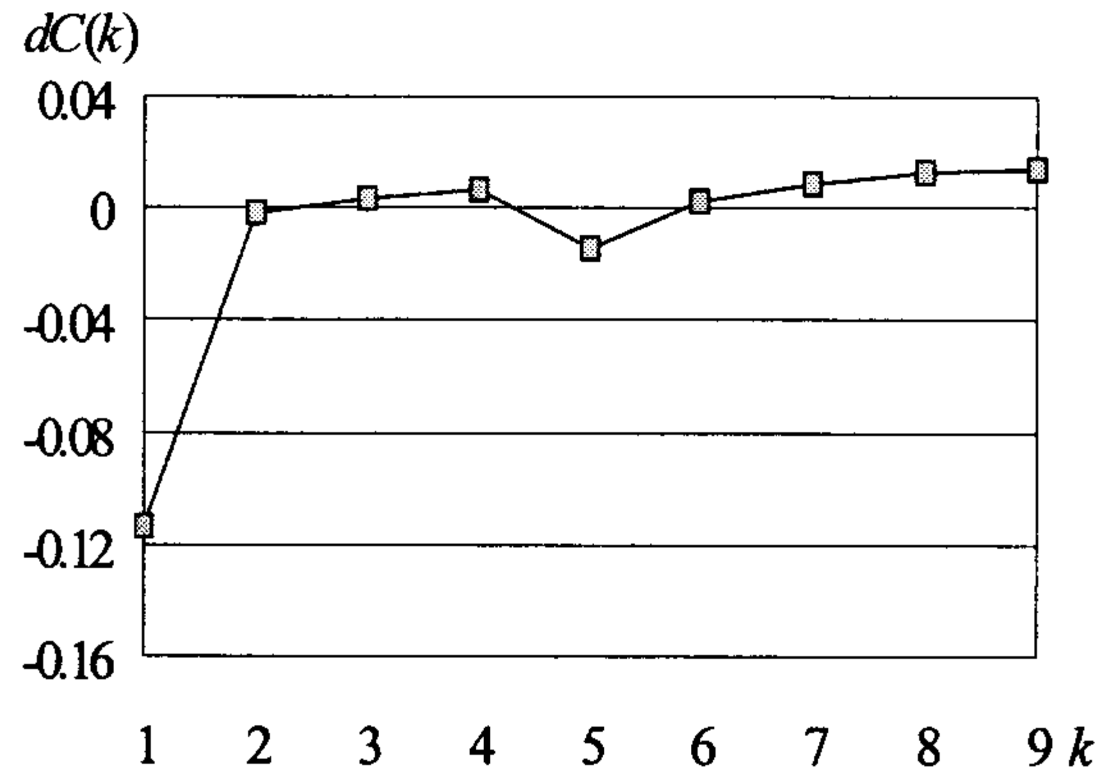


Figure 2. $dC(k)$ vs k (n=10, $\lambda=0.02$, $C_0=5$, and $C_R(k)=1+0.5k$)

3. Numerical Examples

For an example, we consider a linear consecutive-k-out-of-3:F system. The replacement cost per component $C_R(k)$ is assumed to be linearly increasing for k, so that $C_R(k)=C_1+C_2k$. Therefore, the expected cost per unit time when k=1,2,3 can derived as given below:

$$C(1)=3\lambda(C_0+C_1+C_2)$$

$$C(2)=\left(C_0+\frac{7}{3}(C_1+2C_2)\right)/\frac{7}{6\lambda}$$

$$C(3)=(C_0+3(C_1+3C_2))/\frac{11}{6\lambda}.$$

Suppose that $\lambda=0.02$, $C_0=5$, $C_1=1$ and $C_2=2$. The expected costs per unit time are 0.48, 0.2857, 0.2836 for k=1, 2 and 3, respectively. Hence the optimal k is 3.

We now report on computational experiments that evaluate the effect of various parameters on the optimal k. Let us consider the case where $\lambda=0.02$ and $C_1=1$. We solved 540 test problems with n=5,10,20,30,40,50, C_0=1,5,10,20,40,60,80,90,100 and C_2=0.1,0.5,1,2,4,6,8,9,10. Some results are summarized in Table 1.

Table 1. The optimal k ($\lambda=0.02$)

C_0/C_1	C_2/C_1	n			
		5	10	20	50
1	0.1	5	10	15	9
	0.5	4	6	2	2
	1	3	2	2	2
	2	1	1	1	1
	10	1	1	1	1
5	0.1	5	10	16	12
	0.5	5	7	4	3
	1	4	6	3	2
	2	3	2	2	2
	10	1	1	1	1
10	0.1	5	10	17	14
	0.5	5	8	5	4
	1	5	6	3	3
	2	4	3	2	2
	10	2	2	2	2
40	0.1	5	10	20	32
	0.5	5	10	13	6
	1	5	8	12	4
	2	5	7	4	3
	10	3	3	2	2
100	0.1	5	10	20	37
	0.5	5	10	16	11
	1	5	10	14	7
	2	5	9	12	5
	10	4	6	3	3

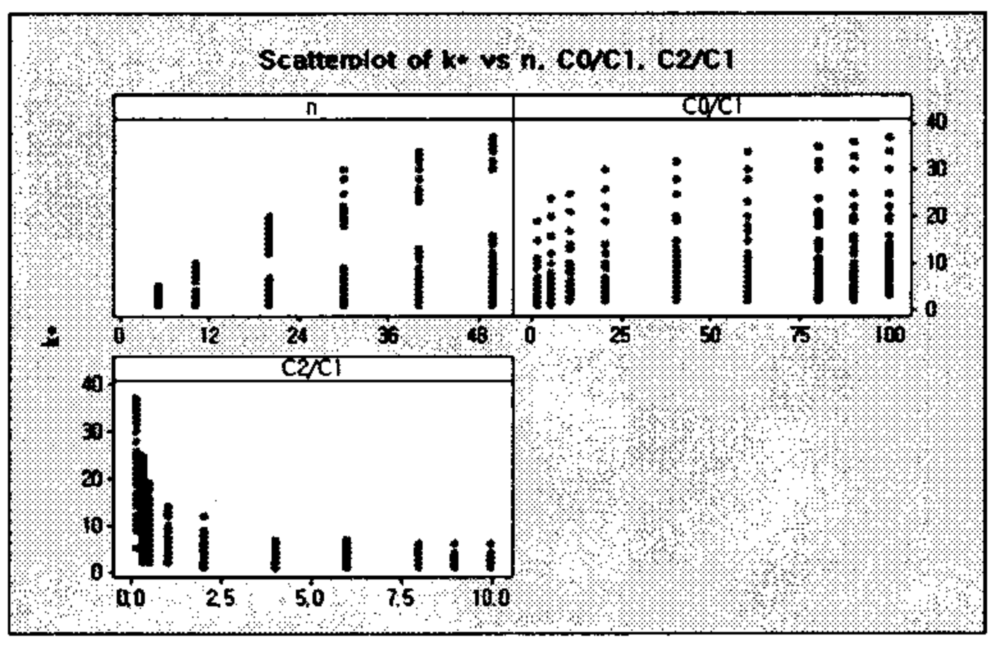


Figure 3. Scatter plots of optimal k vs parameters

Figure 3 shows the scatter diagram of optimal k vs three parameters, where they have fully includes the 540 data that are not depicted in Table1. This figure instinctively shows that the optimal k is affected by all the parameters n, C_0/C_1 and C_2/C_1. Figure 4 depicts that the optimal k increases in C_0/C_1 and decreases in C_2/C_1. In addition, it shows the effect of n is ambiguous. But Figure 4 and Table 1 suggest that the optimal k is n (this means the parallel system is the optimal structure) for the small system and tends to decrease with larger n. To investigate the intersection effect, three intersection plots are depicted in Figure 5. It shows that the intersection between the n and C_2/C_1 is relatively more effective on optimal k.

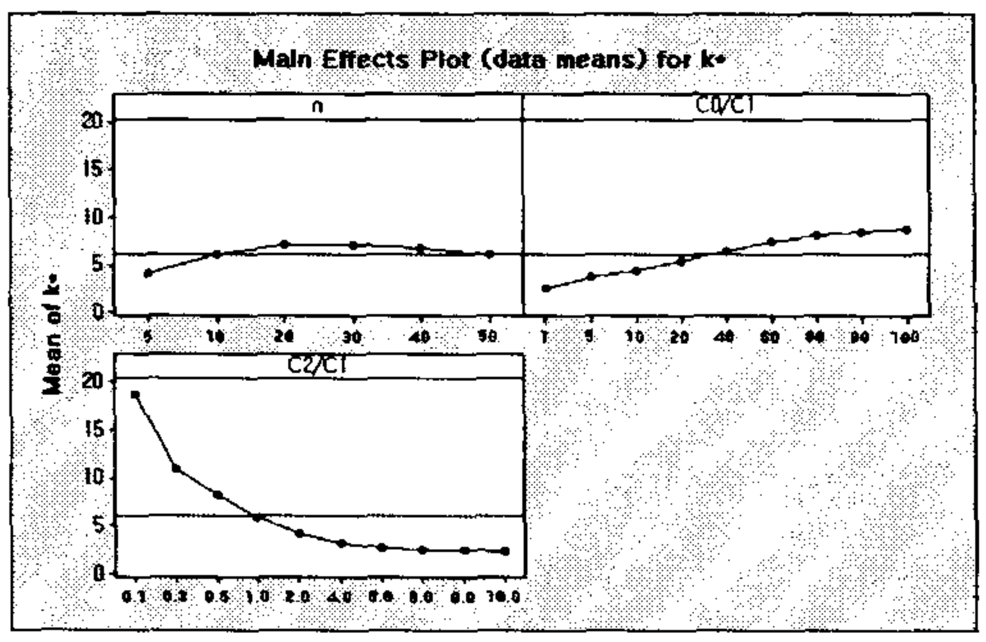
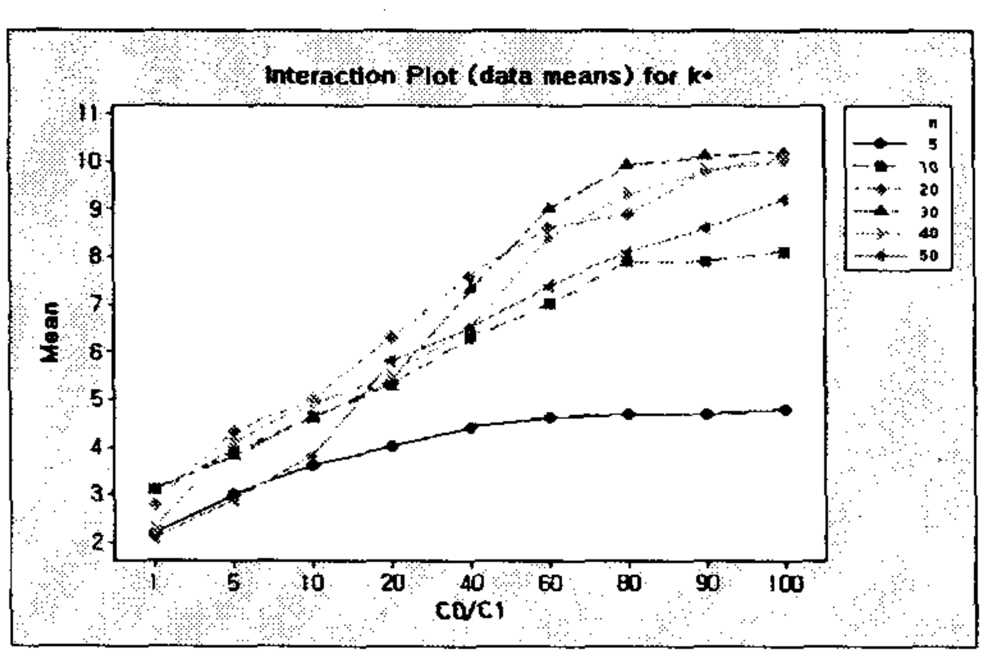
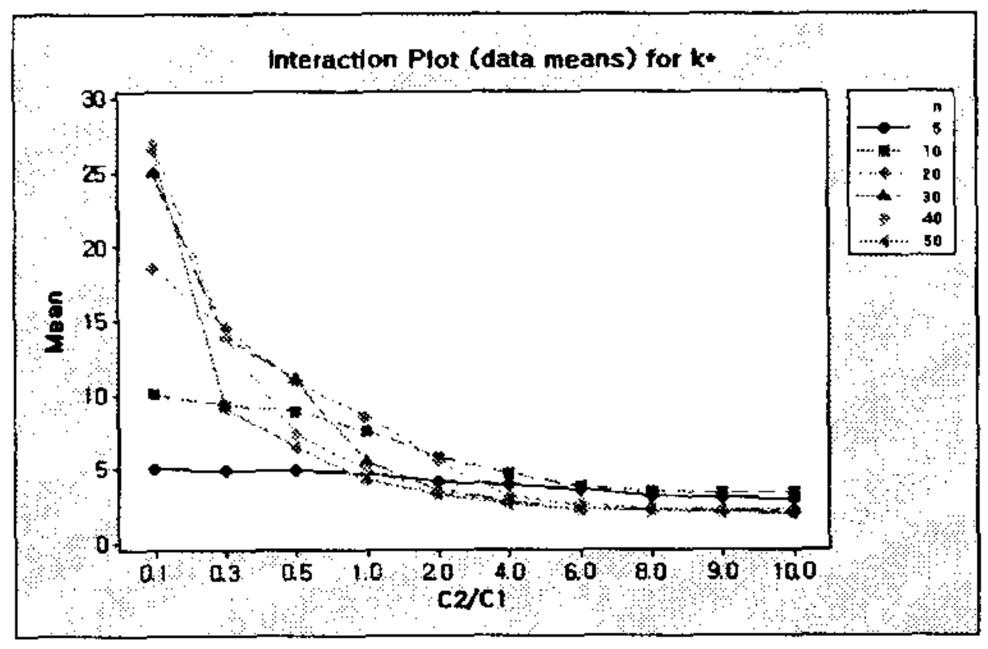


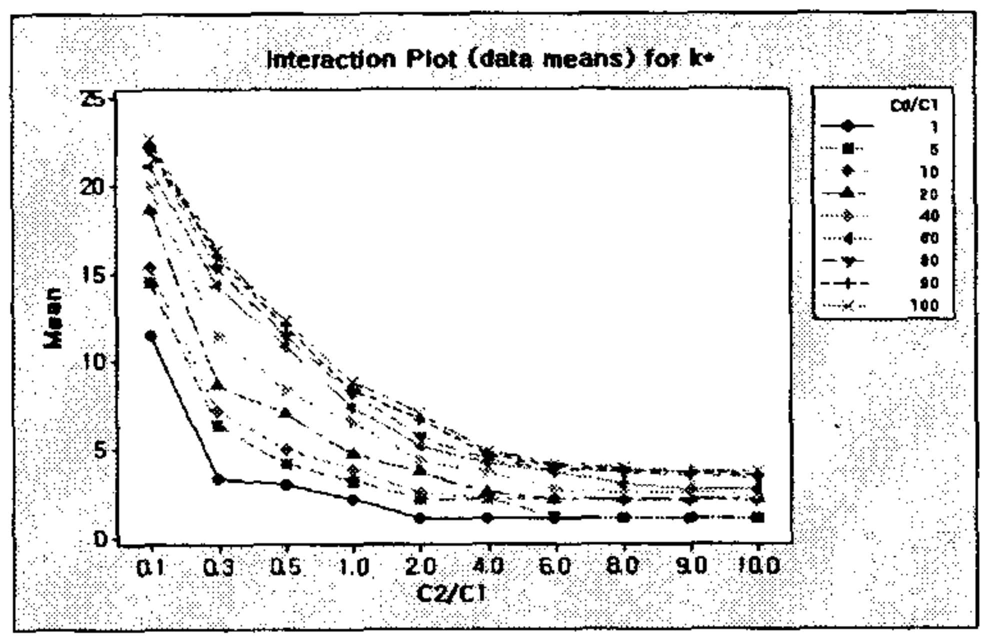
Figure 4. Main effect plots of three parameters for optimal k



(a) Intersection between n and C_0/C_1



(b) Intersection between n and C_2/C_1



(c) Intersection between C_0/C_1 and C_2/C_1

Figure 5. Two-order intersection plots of three parameters.

4. Conclusion

The economic design problem was discussed for the independent model between the component in the linear consecutive-k-out-of-n:F system. For the system with independent components, we derived the expected cost per unit time and determined the optimal system performance k to minimize the expected cost per unit time. From the extensive computational experiment, we found two interesting results that the parallel system is the optimal structure for the small n and tends to decrease with larger n, and that the intersection between the n and C_2/C_1 is more effective on optimal k.

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