The CV Control Chart

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Monitoring variability is a vital part of modern statistical process control. The conventional Shewhart R and S charts address the setting where the in-control process readings have a constant variance. In some settings, however, it is the coefficient of variation, rather than the variance, that should be constant. This paper develops a chart, equivalent to the S chart, for monitoring the coefficient of variation using rational groups of observations.

Key Words: Statistical process control, process variability

Introduction

Shewhart control charts are widely used SPC tools for detecting changes in a quality characteristic of a process and triggering the search for assignable causes of variation. Traditionally, charting was used in manufacturing settings, but its reach has in recent times been extended to education, health care, and a variety of societal applications.

In many settings, while the process is in control, the process readings have a constant mean and variance. In such a setting, the $\overline{X}$ chart is used to monitor the mean, and the R or S control chart is usually used to monitor the variation of the process. However there are some settings in which the mean is not constant, and then SPC control reduces to monitoring the variability only. Our example given below illustrates this setting, and represents a common clinical chemistry control problem in which we make repeat measurements of some characteristic (for example the amount of a chemical in a patient's blood) and want to maintain quality control checks on the laboratory's measurements. As

the mean amount varies from patient to patient, $\overline{X}$ charting is not particularly helpful, but monitoring the variability between the repeat measurements is helpful. As a further complication, in many of these settings, the variance is a function of the mean, making it impossible to use the regular R or S charts for this control. Instead though, we can tailor the technology to take account of the dependence and produce 'pivotals' that measure the inherent variability of the process without the interference of the varying mean.

The remedy depends on the nature of the relationship between the mean and variance of the process readings. One common relationship is that the standard deviation of measurement, σ is directly proportional to the mean μ, so that the 'coefficient of variation' $\gamma=\sigma/\mu$ is a constant. This suggests that we monitor the process by taking rational groups, computing the mean and standard deviation, and chart the sample coefficient of variation

$$W = S / \overline{X}$$

In this paper, we develop the necessary technical underpinnings for this CV (Coefficient of Variation) control chart and illustrate its performance.

Distribution of CV

Starting with the familiar joint distribution of the sample mean and standard deviation of a random sample of size n from a normal $N(\mu,\sigma^2)$.the distribution of the coefficient of variation W can be found by a change of variable and integration. This was done by Hendricks and Robey (1936), giving the density in the form

$$f(v) = k(v)\int_{r=0}^{\infty} r^{n-1}\exp(-\{ar-b(v)\})dr, \text{ where}$$

$$v = \arctan(w),\ a = \sqrt{n}/(\sigma\sqrt{2}),\quad b(v) = a\mu\cos(v),$$

$$k(v) = n^{n/2}\sin^{n-2}(v)\exp(-n\mu^2\sin^2(v)/\{2\sigma^2\})/(2^{n/2-1}\sigma^n\sqrt{\pi}\Gamma\{[n-1]/2\})$$

We develop a canonical form for the distribution of the coefficient of variation. Let $Y \sim N(\gamma^{-1},1)$, and independently $f V^2$ be χ^2_f. Then defining

$$U = \frac{V}{Y}$$

the density function of U is given explicitly as

$$f_1(u) = \frac{A(u)}{(1+u^2)^{(f+1)/2}} \mathrm{I}_f\left(\frac{1}{\gamma(1+u^2)^{0.5}}\right) \quad \text{if } u \geq 0$$

$$f_2(u) = \frac{(-1)^{f-1} A(u)}{(1+u^2)^{(f+1)/2}} \mathrm{I}_f\left(\frac{1}{\gamma(1+u^2)^{0.5}}\right) \quad \text{if } u < 0$$

where the function $A(u)$is given in the Appendix and $\mathrm{I}_f(b) = \int_0^\infty q^f \exp\left[-\frac{1}{2}(q-b)^2\right] dq$ is the Airy function (Iglewicz 1967).

The most familiar, and most common framework, however, is that in which we have a sample of size n from a normal distribution with true mean μ, and with standard deviation σ proportional to the mean: $\sigma=\gamma\mu$. Form the sample mean $\overline{X} \sim N(\mu, (\gamma\mu)^2/n)$ and standard deviation $S \sim \gamma\mu\chi_f/\sqrt{f}$. Then the sample coefficient of variation is

$$W = \frac{S}{\overline{X}}$$

Writing $V=S/(\gamma\mu)$ and $Y = \sqrt{n}\frac{\overline{X}}{\gamma\mu}$, we get $W = \sqrt{n}\frac{V}{Y}$ where V has the distribution of a sample standard deviation with n-1 degrees of freedom, and $Y \sim N[\frac{\sqrt{n}}{\gamma}, 1]$.

Thus the sample coefficient of variation W is $\sqrt{n}$ times a canonical $C(\gamma/\sqrt{n}, n-1)$ variate.

Control Limits for the CV Control Chart

Table 1 provides control limits for the CV chart for a selection of values of n and γ. The probability of exceedance of these limits is 1/740 on each side when the process is in control. For values other than those listed in the tables, interpolation linear in $\sqrt{n}$ should provide adequate accuracy.

Table 1. Control Limits for the Combination of γ and n.

	n=5		n=10		n=15		n=20	
γ	LCL	UCL	LCL	UCL	LCL	UCL	LCL	UCL
0.02	0.00325	0.04222	0.00743	0.03471	0.00957	0.03174	0.01090	0.03005
0.03	0.00488	0.06337	0.01114	0.05210	0.01435	0.04763	0.01634	0.04509
0.04	0.00650	0.08458	0.01485	0.06951	0.01913	0.06355	0.02178	0.06014
0.05	0.00812	0.10587	0.01855	0.08696	0.02391	0.07949	0.02722	0.07522
0.06	0.00975	0.12725	0.02226	0.10446	0.02868	0.09546	0.03266	0.09033
0.07	0.01137	0.14875	0.02595	0.12202	0.03344	0.11148	0.03809	0.10547
0.08	0.01298	0.17038	0.02965	0.13966	0.03820	0.12755	0.04351	0.12065
0.09	0.01460	0.19217	0.03333	0.15737	0.04296	0.14367	0.04892	0.13587
0.10	0.01621	0.21414	0.03701	0.17518	0.04770	0.15986	0.05432	0.15115
0.12	0.01943	0.25869	0.04435	0.21111	0.05716	0.19247	0.06510	0.18188
0.14	0.02264	0.30423	0.05165	0.24754	0.06657	0.22542	0.07583	0.21288
0.16	0.02583	0.35097	0.05892	0.28457	0.07594	0.25879	0.08651	0.24421
0.18	0.02900	0.39912	0.06613	0.32229	0.08525	0.29263	0.09713	0.27591
0.20	0.03215	0.44894	0.07330	0.36081	0.09450	0.32702	0.10768	0.30804
0.22	0.03529	0.50070	0.08042	0.40025	0.10368	0.36203	0.11816	0.34064
0.24	0.03840	0.55474	0.08747	0.44071	0.11279	0.39773	0.12856	0.37377
0.26	0.04148	0.61142	0.09447	0.48233	0.12183	0.43420	0.13888	0.40749
0.28	0.04455	0.67117	0.10140	0.52526	0.13078	0.47152	0.14911	0.44186
0.30	0.04758	0.73449	0.10826	0.56964	0.13965	0.50978	0.15925	0.47694

Example

Following organ transplantation, immunosuppressive drugs are required to prevent rejection of the implanted organ. Cyclosporine is one such widely-adopted drug. For patients undergoing immunosuppressive treatment, it is vital to control the amount of the drug circulating in the blood. Too much weakens the immune system and leads to dangerous infections; too little may lead to organ rejection. The circulating drug level that best balances these conflicting requirements varies widely from patient to patient. For this reason frequent blood assays are required to find the drug level that best stabilizes each

particular patient.

To monitor the precision of a laboratory's assay procedures, quintuplicate measurements are made of blood samples. Concern focuses, not on the means of these assays (which can and do vary widely) but on their variability. Experience has shown that the standard deviation of measurement is approximately proportional to the mean; that is, the coefficient of variation is constant across the range of cyclosporine levels of clinical interest.

We gathered a calibration sample (kindly supplied by DiaSorin Inc.) comprising quintuplicate (n=5) assays on the samples m=35 and plotted the CV control chart in Figure 1.

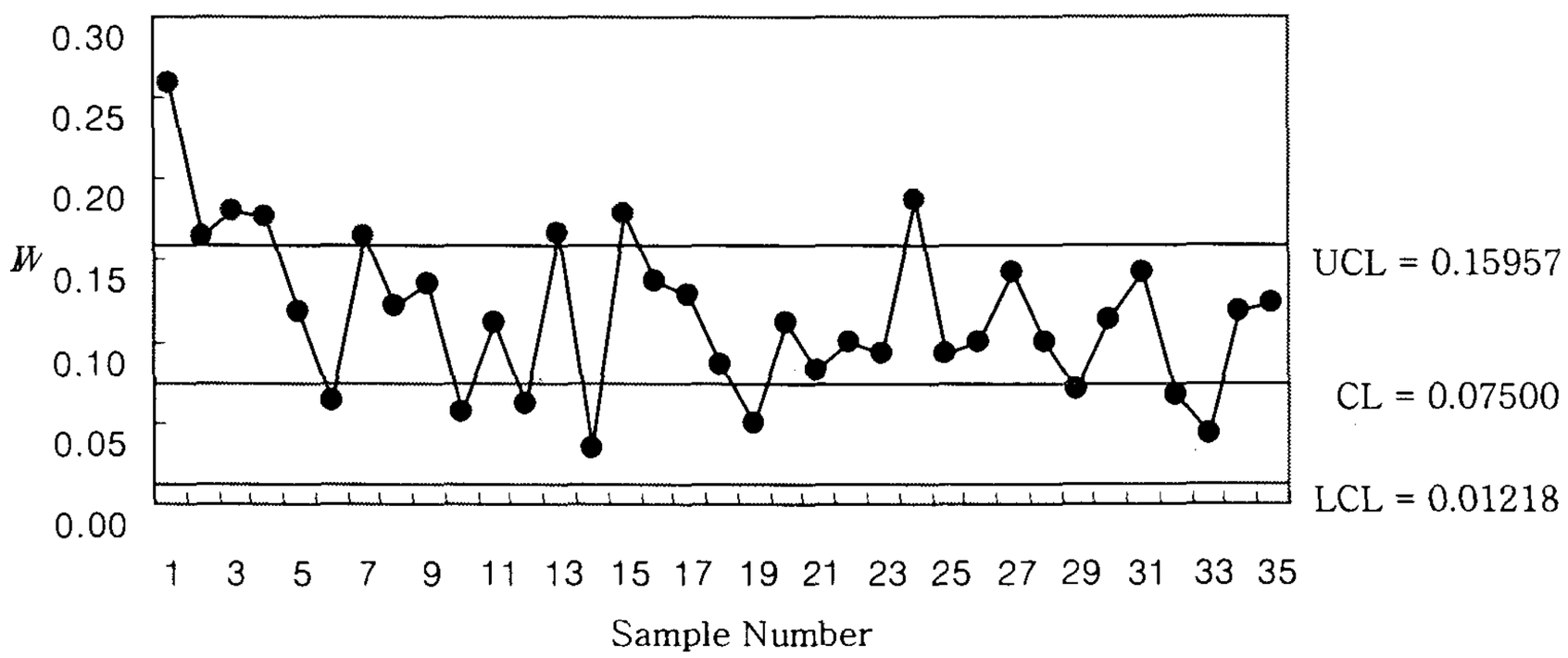

Figure 1 The CV Control Chart of the Fresh Data.

Conclusions

Control charting is an important tool in quality improvement, and, with the increasing awareness of variability as a vital component of quality, charting for variability has assumed a greater importance. The traditional $\overline{X}$ and S charts are not suitable for the situation, common in clinical chemistry and some other settings, where neither the process mean nor the variance is constant, but the coefficient of variation is. For this setting, a chart of the coefficient of variation is a potentially attractive tool. We have developed the methodology for such a chart and shown its promising performance.

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