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구간치 쇼케이적분에의해 정의된 집합체 연산자의 성질

Some properties of interval-valued Choquet integral-baed aggregation operators

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요 약

본 논문은 집합치 집합체 연산자를 정의하고 이들의 성질들을 조사한다. 또한 구간치 쇼케이 적분에의해 정의된 집합체 연산자를 정의 하고 이들의 특성들을 제시한다.

Abstract

In this paper, we consider set-valued aggregation operators and investigate some properties of them. Moreover, we define interval-valued Choquet integral aggregation operators and discuss their characterizations.

Key Words : Aggregation operator, Interval-valued random variable, Choquet integral, Fuzzy measure

1. Preliminaries and Definitions

Let X be a non-empty index set and $\mathfrak{I}$ a σ-algebra of subsets of X. A set function $\mu: \mathfrak{I} \to [0,1]$ is called a fuzzy measure if it is monotone and $\mu(\emptyset)=0$(see [3,4,14,16,17,18]).

If X is a finite set and $\mathfrak{I}=2^X$, a fuzzy measure $\mu: \mathfrak{I} \to [0,1]$ is called symmetric if it is invariant under bijective transformations of X, i.e., for each bijection $\phi: X \to X$ and for each $E \in \mathfrak{I}$ we have $\mu(\phi^{-1}(E))=\mu(E)$.

For each set $B \subset R^2$, let $\Sigma(B)$ be the σ-algebra of all Borel subsets of B. Denoted by $L(\mathfrak{I})$ the set of all random variables which are $\mathfrak{I}$-measurable functions from X to $[0,1]$.

In previous works [5,13], the authors investigated measure-based aggregation operators and Choquet integral-based aggregation operators. These constructions are useful methods to decision making, information theory, expected utility theory, and risk analysis.

We note that set-valued Choquet integrals were first introduced by Jang, Kil, Kim and Kwon[6] and restudied by Zhang, Guo and Lia[19]. In the papers([6,7,8,9,10,11,19]), they have been studied some properties of set-valued Choquet integrals and structural characteristics of interval-valued Choquet integrals.

Definition 1.1 ([3,4,14,15,16,17,18]) (1) The Choquet integral of a random variable f with respect to a fuzzy measure μ on $A \in \mathfrak{I}$ is defined by

$$(C)\int_A f\,d\mu = \int_0^\infty \mu(\{x|f(x)>\alpha\}\cap A)\,d\alpha$$

where the integrand on the right-hand side is an ordinary one.

(2) A random variable f is called c-integrable if the Choquet integral of f can be defined and its value is finite.

We note that the Choquet integral is a generalization of the Lebesgue integral, since they coincide when μ is a classical(σ

-additive) measure.

Definition 1.2 ([3,4,14]) Let f, g be nonnegative random variables. We say that f and g are comonotonic, in symbol $f \sim g$ if

$$f(x) < f(x') \Rightarrow g(x) \le g(x') \text{ for all } x, x' \in X$$

Theorem 1.3 ([3,4,14]) Let f, g be nonnegative random variables. Then we have

(1) $f \sim f$,

(2) $f \sim g \Rightarrow g \sim f$,

(3) $f \sim a$ for all $a \in R^+$,

(4) $f \sim g$ and $g \sim h \Rightarrow f \sim (g+h)$.

Theorem 1.4 ([3,4,14]) Let f, g be nonnegative random variables. Then we have the followings.

(1) If $f \le g$, then $(C)\int f d\mu \le (C)\int g d\mu$.

(2) If $A \subset B$ and $A, B \in \mathfrak{I}$, then

$$(C)\int_A f d\mu \le (C)\int_B f d\mu.$$

(3) If $f \sim g$ and $a, b \in R^+$, then

$$(C)\int (af+bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu$$

(4) If $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$, then

$$(C)\int f \vee g d\mu \ge (C)\int f d\mu \vee (C)\int g d\mu$$

and

$$(C)\int f \wedge g d\mu \le (C)\int f d\mu \wedge (C)\int g d\mu.$$

We denote $I([0,1])$ by

$$I([0,1]) = \overline{a} = [a^-, a^+] \mid a^- \le a^+, a^-, a^+ \in [0,1]$$

For any $a \in [0,1]$, we define $a = [a, a]$. Obviously, $a \in I([0,1])$.

Definition 1.5 For two intervals $\overline{a} = [a^-, a^+], \overline{b} = [b^-, b^+] \in I([0,1])$, then we define

(1) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,

(2) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$

(3) $\overline{a} \le \overline{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$,

(4) $\overline{a} < \overline{b}$ if and only if $\overline{a} \le \overline{b}$ and $\overline{a} \ne \overline{b}$,

(5) $\overline{a} \subset \overline{b}$ if and only if $b^- \le a^- \le a^+ \le b^+$.

We remark that $I([0,1])$ satisfies idempotent law, commutative law, associative law, absorption law, and distributive law.

Theorem 1.7 (1) If we define

$$\overline{a} \cdot \overline{b} = \{x \cdot y \mid x \in \overline{a}, y \in \overline{b}\}$$

for all $\overline{a}, \overline{b} \in I([0,1])$, then we have

$$\overline{a} \cdot \overline{b} = [a^- b^-, a^+ b^+].$$

(2) If $d_H : I([0,1]) \times I([0,1]) \to [0, \infty)$ is a Hausdorff metric defined by

$$d_H(A, B) = max\{sup_{x \in A} inf_{y \in B} |x-y|, sup_{y \in B} inf_{x \in A} |x-y|\}$$

then we have

$$d_H(\overline{a}, \overline{b}) = max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Let $\overline{L}(\mathfrak{I})$ theset of all interval-valued random variables which are $\mathfrak{I}$-measurable interval-valued functions from X to $I([0,1]) \setminus \{\emptyset\}$. We recall that an interval-valued function $\overline{f} \in \overline{L}(\mathfrak{I})$ is $\mathfrak{I}$-measurable if for any open set $O \subset [0,1]$,

$$\overline{f}^{-1}(O) = \{x \in X \mid \overline{f}(x) \cap O \ne \emptyset\} \in \mathfrak{I}.$$

Definition 1.8 Let $\{A_n\} \subset I([0,1])$ be a sequence and $A \in I([0,1])$. We define

(1) $A_n \uparrow (\downarrow) A(order)$ if $d_n(A_n, A) \to 0$ as $n \to \infty$ and $A_n \le (\ge) A_{n+1}$ for all $n = 1, 2, \cdots$,

(2) $A_n \uparrow (\downarrow) A(inclusion)$ if $d_n(A_n, A) \to 0$ as $n \to \infty$ and $A_n \subset (\supset) A_{n+1}$ for all $n = 1, 2, \cdots$,

Definition 1.9 (1) The Choquet integral of an interval-valued random variable $\overline{f}$ on $A \in \mathfrak{I}$ is defined by

$$(C)\int_A \overline{f} d\mu = \{(C)\int_A f d\mu \mid f \in S(\overline{f})\}$$

where $S(\overline{f})$ is the family of measurable selections of $\bar{f}$.

(2) $\bar{f}$ is said to be C-integrable if

$$(C)\int \bar{f} d\mu \neq \emptyset.$$

(3) $\bar{f}$ is said to be Choquet inetgrably bounded if there is a C-integrable random variable g such that

$$\|\bar{f}\| = sup_{r \in \bar{f}(x)} |r| \leq g(x), \text{ for all } x \in X.$$

Instead of $(C)\int_X \bar{f} d\mu$, we write $(C)\int \bar{f} d\mu$. We recall that if $A, B \in C(X)$(the class of closed subsets of X), then $A \leq B$ means

$$infA \leq infB \text{ and } supA \leq supB.$$

Theorem 1.10 ([12, 19]) (1) If a closed set-valued random variable $\bar{f}$ is C-integrable and if $A \leq (\subset) B$ and $A, B \in C(X)$, then

$$(C)\int_A \bar{f} d\mu \leq (\subset)(C)\int_B \bar{f} d\mu.$$

(2) If a fuzzy measure μ is continuous and a closed set-valued random variable $\bar{f}$ is Choquet integrably bounded, then $(C)\int \bar{f} d\mu$ is a closed set.

(3) If a fuzzy measure μ is continuous and an interval-avlued random variable $\bar{f} = [f^-, f^+]$ is Choquet integrably bounded, then we have

$$(C)\int \bar{f} d\mu = [(C)\int f^- d\mu, (C)\int f^+ d\mu].$$

2. Set-valued aggregation operators.

If $f \in L(\Im)$, we define the function $h_{\mu,f}: [0,1] \to [0,1]$ by

$$h_{\mu,f}(t) = \mu(\{f \geq t\}).$$

and the interval-valued function $\bar{h}_{\mu,\bar{f}}: [0,1] \to I([0,1])$ by

$$\bar{h}_{\mu,\bar{f}}(t) = \{h_{\mu,f}(t) | f \in S(\bar{f})\}$$

Then we obtain some basic properties of $\bar{h}_{\mu,\bar{f}}$

Theorem 2.1 If $\bar{f}$ is an interval-valued random variable and if μ is continuous, then the function $\bar{h}_{\mu,\bar{f}}$ is non-increasing Borel measurable and $\bar{h}_{\mu,\bar{f}}(0) = 1$

Definition 2.2 ([13]) Let $(X, \Im, \mu)$ and $((0,1)^2, B(0,1)^2, m)$ be two fuzzy measure spaces. The functional $M_{\mu,m}: L(\Im) \to [0,1]$ given by

$$M_{\mu,f}(f) = m(\{(x,y) \in (0,1)^2 | y < h_{\mu,f}(x)\})$$

is called (μ, m)-aggregation operator.

Definition 2.3 The interval-valued functional $\bar{M}_{\mu,m}: \bar{L}(\Im) \to \wp([0,1])$ given by

$$\bar{M}_{\mu,m}(\bar{f}) = \{M_{\mu,m}(f) | f \in S(\bar{f})\}$$

is called a set-valued (μ, m)-aggregation operator.

Obviously, we have the following basic properties.

Theorem 2.4 Let μ, m be two fuzzy measures as in Definition 2.2. If is a set-valued (μ, m)-aggregation operator, then we have

(1) $\bar{M}_{\mu,m}(0) = 0$ and $\bar{M}_{\mu,m}(1) = 1$,

(2) If $\bar{f}, \bar{g} \in \bar{L}(\Im)$ and $\bar{f} \leq \bar{g}$, then
$$\bar{M}_{\mu,m}(\bar{f}) \leq \bar{M}_{\mu,m}(\bar{g}).$$

Now, we investigate some characterizations of set-valued (μ, m)-aggregation operators. We denote that a crisp subset E of X $I_E: X \to \{0,1\}$ is the characteristic function of E mapping exactly the elements of E to 1.

Theorem 2.5 Let $(X, \Im, \mu)$ and $((0,1)^2, B((0,1)^2), m)$ be two fuzzy measures and $\bar{M}_{\mu,m}: \bar{L}(\Im) \to \wp([0,1])$ a set-valued (μ, m)-aggregation operator . Then we have the followings;

(1) $\bar{M}_{\mu,m}$ is idempotent for each fuzzy measure μ on $(X, \Im)$ if and only if for all $x \in (0,1)$, we have
$$m((0,x) \times (0,1)) = x.$$

(2) The fuzzy measure μ on $(X, \Im)$ can be reproduced from $\bar{M}_{\mu,m}$ via $\mu(E) = \bar{M}_{\mu,m}(I_E)$ if and only if
$$m((0,1) \times (0,x)) = x$$
for all $x \in Ran(\mu)$.

(3) If X is a finite set and if a fuzzy measure μ is symmetric, then $\bar{M}_{\mu,m}$ is symmetric for each fuzzy measure m on $((0,1)^2, B(0,1)^2)$.

3. Interval-valued Choquet integral-based aggregation operators.

In this section, using Corollary 3.4 ([13]), we discuss interval-valued Choquet integral-based aggregation operators. We recall that especially important is the case when we are constructing an (μ, m)-aggregation operator by means of some σ-additive $m B((0,1)^2) \to [0,1]$, in which case m is a probability measure on the product space $((0,1)^2, B((0,1)^2))$. By Klement et al. ([13, p.9]), there exists a copula $C[0,1]^2 \to [0,1]$ such that for all $(x,y) \in (0,1)^2$,

$$m((0,x)\times(0,y)) = C(P_1((0,x)), P_2((0,y)))$$

where P_1 and P_2 are the respective marginal probabilities of m.

Let me introduce the following theorem as a consequence of Klement et al.'s Corollary 4.1([13]).

Theorem 3.1 Let $m: B((0,1)^2) \to [0,1]$ be a probability measure with marginal $P_1 = \lambda$ (the Lebesgue measure) and $P_2 = P$ linked by the product copula T_p i.e.

$$m((0,x)\times(0,y)) = xP((0,y))$$

for all $(x,y) \in (0,1)^2$. Let $(X, \mathfrak{I}, \mu)$ be a fuzzy measure space such that the underlying topological space $(X, \mathfrak{I})$ is compact. Then the (μ, m)-aggregation operator $M_{\mu,m}$ is comonotonic additive, i.e. for all comonotonic random variables $f, g: X \to [0,1]$ with $f+g \le 1$,

$$M_{\mu,m}(f+g) = M_{\mu,m}(f) + M_{\mu,m}(g).$$

Keeping all the notations and hypotheses of Theorem 3.1 and introducing another fuzzy measure $P \circ \mu: \mathfrak{I} \to [0,1]$ by

$$P \circ \mu(E) = P((0, \mu(E))),$$

it is well-known that the (μ, m)-aggregation operator $M_{\mu,m}$ discussed in Theorem 3.1 equals the Choquet integral with respect to $P \circ \mu$ (see [13, p.9]), i.e.

$$M_{\mu,m}(f) = (C)\int_X f dP \circ \mu \qquad (3.1)$$

for all $f \in L(\mathfrak{I})$. Thus we can obtain the following theorem.

Theorem 3.2 Let $\mu, m, P_1 = \lambda, P_2 = P$ be as in Theorem 3.1. Then under the same hypotheses of Theorem 3.1, $\overline{M}_{\mu,m}$ is represented by

$$\overline{M}_{\mu,m}(\overline{f}) = (C)\int \overline{f} dP \circ \mu.$$

Theorem 3.3 Let $\mu, m, P_1 = \lambda, P_2 = P$ be as in Theorem 3.1 and suppose the same hypotheses of Theorem 3.1. If a fuzzy measure μ is continuous and $\overline{f}$ is Choquet integrably bounded, then $\overline{M}_{\mu,m}(\overline{f})$ is interval in $I([0,1])$.

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