A common fixed point theorem in the intuitionistic fuzzy metric space

직관적 퍼지거리공간에서 공통 부동점 정리

Jong Seo Park*, Seon Yu Kim* and Hong Jae Kang*

* Department of Mathematics Education, Chinju National University of Education, Jinju, 660-756, Korea

Abstract

The purpose of this paper is to establish the common fixed point theorem in the intuitionistic fuzzy metric space in which it is a little revised in Park [11]. Our research are an extension of Jungck's common fixed point theorem [8] in the intuitionistic fuzzy metric space.

Key Words: Common Fixed Point, Intuitionistic Fuzzy Metric Space

1. Introduction Preliminaries

Zadeh [17] was introduced to the concept of fuzzy sets, Lowen [10] is defined convergence in a fuzzy topological space which enables us to characterize fuzzy compactness. Grabiec [6], Park and Kim [12] are studied a fixed point in a fuzzy metric space introduced by Kramosil and Michalek [9], and Subrahmanyam [16] is proved a common fixed point theorem in fuzzy metric spaces.

On the other hand, Attanassov [1] generalized this idea to intuitionistic fuzzy sets, and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [1-4, 11]. Also, Park [11] is defined an intuitionistic fuzzy metric space, and Park, Kwun and Park [13] are studied a fixed point theorem in an intuitionistic fuzzy metric space.

In this note, Jungck's common fixed point theorem in metric space is generalized in this intuitionistic fuzzy metric space in which it is a little revised in Park [11]..

2. Preliminaries

Now, we will give some definitions, properties and notation of the intuitionistic fuzzy metric space.

Definition 2.1([15]). A binary operation $*:[0,1\times[0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative,

(b) $*$ is continuous,

(c) $a*1=a$ for all $a\in[0,1]$,

(d) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ $(a,b,c,d\in[0,1])$.

Definition 2.2([15]). A binary operation $\diamond$ $:[0,1]\times[0,1] \rightarrow [0,1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions:

(a) $\diamond$ is commutative and associative,

(b) $\diamond$ is continuous,

(c) $a\diamond 1=a$ for all $a\in[0,1]$,

(d) $a\diamond b\geq c\diamond d$ whenever $a\leq c$ and $b\leq d$ $(a,b,c,d\in[0,1])$.

Definition 2.3. The 5-tuple $(X, M, N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions; for all $x,y,z\in X$, such that

(a) $M(x,y,t)>0$,

(b) $M(x,y,t)=1 \leftrightarrow x=y$,

(c) $M(x,y,t)=M(y,x,t)$,

(d) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,

(e) $M(x,y,\cdot): (0,\infty)\rightarrow(0,1]$ is continuous,

(f) $N(x,y,t)>0$,

(g) $N(x,y,t)=0 \leftrightarrow x=y$,

(h) $N(x,y,t)=N(y,x,t)$,

(i) $N(x,y,t)\diamond N(y,z,s)\geq N(x,z,t+s)$,

(j) $N(x,y,\cdot): (0,\infty)\rightarrow(0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x,y,t) and N(x,y,t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

In this note, we shall denote the intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ by X.

Lemma 2.1([6], [12]). In an intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is nondecreasing and $N(x,y,\cdot)$ is nonincreasing for all $x,y\in X$.

In all that follows **N** stands for the set of natural numbers and X stands for an intuitionistic fuzzy metric space X with the following properties:

(2.1) $\lim_{t\to\infty} M(x,y,t)=1, \quad \lim_{t\to\infty} N(x,y,t)=0$ for all $x,y\in X$

Lemma 2.2([13]). Let X be an intuition -istic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the intuitionistic fuzzy metric. Then for a sequence $\{x_n\}\subset X$, $x_n\rightarrow x$ if and only if $M(x_n,x,t)\rightarrow 1$ and $N(x_n,x,t)\rightarrow 0$ as $n\rightarrow\infty$.

Definition 2.4. Let X be an intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$ in a intuitionistic fuzzy metric space X is called Cauchy if $\lim_{t\to\infty} M(x_{n+p}, x_n ,t)=1$, $\lim_{t\to\infty} N(x_{n+p}, x_n ,t)=0$ for every t>0 and each p>0.

(b) X is complete if every Cauchy sequence in X converges in X.

(c) A sequence $\{x_n\}$ in X is convergent to $x\in X$ if $\lim_{t\to\infty} M(x_n, x, t)=1$$, $\lim_{t\to\infty} N(x_n, x, t)=0$ for each t>0.

(d) A map $f: X\rightarrow X$ is called continuous at x_0 if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converging to x_0.

Lemma 2.3. If $\{x_n\}$ is a sequence in an intuitionistic fuzzy metric space X and $M(x_n, x_{n+1}, t)\geq M(x_0, x_1, \frac{t}{\alpha^n})$, $N(x_n, x_{n+1}, t)\leq N(x_0, x_1, \frac{t}{\alpha^n})$ where α is a positive number with $0<\alpha<1$ and $n=1,2,\cdot$, $s*s\geq s$, $r\diamond r\leq r$ for $s,r\in[0,1]$, then $\{x_n\}$ is a Cauchy sequence in X

3. Results

The following theorem has a intuitionistic fuzzy analogue for Jungck's [8].

Theorem 3.1. Let X be a complete intuitionistic fuzzy metric space and let $f, g: X\rightarrow X$ be maps that satisfy the following conditions:

(a) $g(X)\subseteq f(X)$.

(b) f is continuous.

(c) M and N are satisfied the following conditions:

(3.1) $M(g(x),g(y),\alpha t)\geq M(f(x),f(y),t)$,
$N(g(x),g(y),\alpha t)\leq N(f(x),f(y),t)$

for all $x,y\in n$, $t>0$ and $0<\alpha<1$.

Then f and g have a unique common fixed point provided f and g commute.

Proof. Let x_0 \in X$. By condition (a), we can find x_1

such that $f(x_1)=g(x_0)$. Therefore we can define a sequence

$\{x_n\}\subset X$ such that $f(x_n)=g(x_{n-1})$ by induction. Also,

$$M(f(x_n),f(x_{n+1}),t)\geq M(f(x_{n-1}),f(x_n),\frac{t}{\alpha})$$

$$\text{(3.2)}\qquad \cdots\cdots\cdots$$

$$\geq M(f(x_0),\ f(x_1),\ \frac{t}{\alpha^n})$$

and

$$N(f(x_n),f(x_{n+1}),t)\leq M(f(x_{n-1}),f(x_n),\frac{t}{\alpha})$$

$$\text{(3.3)}\qquad \cdots\cdots\cdots$$

$$\leq M(f(x_0),\ f(x_1),\ \frac{t}{\alpha^n})$$

So for any positive integer p,

$$M(f(x_n),f(x_{n+p}),t)$$
$$\geq M(f(x_0),f(x_1),\frac{t}{p\alpha^n})$$
$$*\cdots\cdots\cdots\cdots *M(f(x_0),f(x_1),\frac{t}{p\alpha^{n+p-1}})$$

and

$$N(f(x_n),f(x_{n+p}),t)$$
$$\leq N(f(x_0),f(x_1),\frac{t}{p\alpha^n})$$
$$\diamond\cdots\cdots\cdots\cdots \diamond N(f(x_0),f(x_1),\frac{t}{p\alpha^{n+p-1}})$$

By (2.1), since

$$\lim_{t\to\infty} M(f(x_0),f(x_1),\frac{t}{p\alpha^n})=1,$$

$$\lim_{t\to\infty} N(f(x_0),f(x_1),\frac{t}{p\alpha^n})=0,$$

from (3.2) and (3.3),

$$\lim_{t\to\infty} M(f(x_n),f(x_{n+p}),t)\geq 1*\cdots\cdots\cdots\cdots *\geq 1,$$

and

$$\lim_{t\to\infty} N(f(x_n),f(x_{n+p}),t)\leq 0\diamond\cdots\cdots\cdots\cdots\diamond 0\leq 0$$

Therefore by Definition 2.4 and Lemma 2.3, $\{f(x_n)\}$ is Cauchy sequence. By the completeness of X in assumption, there exist $w\in X$ such that $\{f(x_n)\}$ converges to w. So $g(x_{n-1})=f(x_n)$ tends to w as $n\to\infty$. It can be seen from the condition (b) of theorem that the continuity of f implies that of g.

So, $\{g(f(x_n))\}\to g(w)$. However, $g(f(x_n))=f(g(x_n))$ from commutativity of f and g. Hence $f(g(x_n))$ converges to f(w). Since the limits are unique, $f(w)=g(w)$. Also, $f(f(w))=f(g(w))$ by commutativity and

$$M(g(w),g(g(w)),t)\geq M(f(w),\ f(g(w)),\frac{t}{\alpha})$$

$$\text{(3.4)}\qquad \cdots\cdots\cdots$$

$$\geq M(g(w),\ g(g(w)),\frac{t}{\alpha^n})$$

and

$$N(g(w),g(g(w)),t)\leq N(f(w),\ f(g(w)),\frac{t}{\alpha})$$

$$\text{(3.5)}\qquad \cdots\cdots\cdots$$

$$\leq N(g(w),\ g(g(w)),\frac{t}{\alpha^n})$$

By Definition 2.3, (2.1), (3.4) and (3.5), $M(g(w),\ g(g(w)),t)=1$, $N(g(w),\ g(g(w)),t)=0$. Hence $g(w)=g(g(w))$, and $g(w)=g(g(w))=f(g(w))$. Therefore g(w)is a common fixed point of f and g.

If x, z are two fixed points common to f and g, then

$$1\geq M(x,z,t)=M(g(x),g(z),t)$$
$$\geq M(f(x),f(z),\frac{t}{\alpha})=M(x,z,\frac{t}{\alpha})\cdots\cdots\cdots$$
$$\geq M(x,z,\frac{t}{\alpha^n})\to 1 \text{ as } n\to\infty,$$

$$0\leq N(x,z,t)=N(g(x),g(z),t)$$
$$\leq N(f(x),f(z),\frac{t}{\alpha})=N(x,z,\frac{t}{\alpha})\cdots\cdots\cdots$$
$$\leq N(x,z,\frac{t}{\alpha^n})\to 0 \text{ as } n\to\infty.$$

Therefore x=z by Definition 2.3.

Appendix(Jungck's Theorem). Let f be a continuous mapping of a complete metric space

(X,d) into itself and let $g:X \rightarrow X$ be a map. If

(a) $g(X) \subseteq f(X)$,

(b) g commutes with f,

(c) $d(g(x),g(y)) \leq \alpha d(f(x),f(y))$ for some $\alpha \in (0,1)$ and all x and y in X. Then f and g have a unique common fixed point.

4. References

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*E-mail address : parkjs@cue.ac.kr