정규 크리깅보간법을 이용한 응력특이문제의 *p*-적응적 유한요소해석

p-Adaptive Finite Element Analysis of Stress Singularity Problems by Ordinary Kriging Interpolation

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ABSTRACT

This paper is to examine the applicability of ordinary Kriging interpolation(OK) to the *p*-adaptivity of the finite element analysis that is based on variogram. In the *p*-refinement, the analytical domain has to be refined automatically to obtain an acceptable level of accuracy by increasing the *p*-level non-uniformly or selectively. In case of non-uniform *p*-distribution, the continuity between elements with different polynomial orders is achieved by assigning zero higher-order derivatives associated with the edge in common with the lower-order derivatives. It is demonstrated that the validity of the proposed approach by analyzing results for stress singularity problem.

Keywords: Kriging Interpolation, p-refinement, variogram, stress singularity

1. Introduction

In the finite element method, the interpolation is important and fundamental to graphing, analyzing and understanding of 2D data. The Kriging interpolation technique is a method of interpolation which predicts unknown values from data observed at known locations(Stein, 1999). This method uses variogram to express the spatial variation, and it minimizes the error of predicted values which are estimated by spatial distribution of the predicted values. Kriging is a form of weighted average estimator. The weight factors to interpolate the spatial data at arbitrary locations are assigned on the basis of a model fitted to a function, such as the variogram, which represents spatial structure in the variable of interest.

In this study, the weighted least-square is applied to obtain the estimated exact solution from the stress data at the Gauss points. The weight factor is determined by experimental and theoretical variogram for interpolation of stress data apart from the conventional interpolation methods including the least-square fitting that use an equal weight factor.

The adaptive finite element analysis consists of two stages; *a posteriori* error estimate and the mesh refinement. The goal is to refine the mesh so that the error is within the specified tolerance and is as uniformly distributed throughout the domain as possible. The estimated errors in the finite element solution are of primary importance. To minimize the computational cost, an effective and reliable technique of post-processing is necessary for use in adaptive mesh refinement. Two types of *a posteriori* error estimates, namely the stress recovery procedure and the residual technique, may be considered. Stress recovery procedures can be classified as local (*i.e.* element level), patch-based, and global(Yazdani & Riggs, 2000). A recent patch-based procedure is the Superconvergent Patch Recovery (S.P.R.) technique introduced by Zienkiewicz and Zhu (or *Z/Z*)(1992) and subsequently refined by others(Boroomand & Zienkiewicz, 1997;

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Wiberg). It is known that the Z/Z error estimate has not been applied to the p-refinementbecause the shape functions used to interpolate displacements within an element are also used to interpolate recovered stresses.

The objective of this study is to demonstrate the applicability of ordinary Kriging interpolation(OK) to the p-adaptivity of the finite element analysis of stress singularity problem employing the modified S.P.R. method that can recover stresses by using the weighted least-square method. To verify this method, the limit value technique has been proposed as a different error estimator that is based on the exact strain energy when the number of degrees of freedom approaches infinite.

2. Ordinary Kriging Interpolation

The principal tool of most geostatistical analyses is the variogram. Thus the spatial continuity of the data can be examined on the basis of the variogram analysis. The variogram $2\gamma(h)$ is defined as :

$$2\gamma(h) = Var\left[Z(x+h) - Z(x)\right] \tag{1}$$

The $\gamma(h)$ is often called a *semi-variogram* that can be defined as half the expected squared difference between paired random functions separated by the distance and direction vector or *lag*(or *separation distance*) h such as :

$$\gamma(h) = \frac{1}{2n}\sum_{i=1}^{n}\left[z(x_i) - z(x_i+h)\right]^2 \tag{2}$$

where n is equal to the number of pairs of values in which the separation distance is equal to h. The increase of the separation distance cannot cause the semi-variogram increase. The separation distance that causes the semi-variogram reach plateau is called *range* a. Ordinary Kriging(OK) has been used extensively as an estimation technique because of its simplicity and for its reliable estimates. This technique allows estimation of an unsampled location based on neighbor data values(Lloyd & Atkinson, 2001):

$$Z^*(x_o) = \sum_{i=1}^{n}\lambda_i Z(x_i) \tag{3}$$

where $\lambda_i$and $Z(x_i)$are the weights assigned to the available observations and neighbor data close to the unsampled location (x_o), respectively. With OK the weights sum to one to ensure that the estimate is unbiased :

$$1 - \sum_{i=1}^{n}\lambda_i = 0 \tag{4}$$

The Kriging variance associated to an OK estimate is called the minimum variance unbiased estimator(MVUE) or best linear unbiased estimator(BLUE) since the constraint condition defined in Eq.(4) should be applied to minimize the variance of estimate errors. Thus the mathematical equation can be formulated by Eq.(5).

Minimize

$$\sigma_{OK}^2 = \sigma^2 - 2\sum_{i=1}^{n}\lambda_i\sigma_{oi}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\sigma_{ij}^2 \quad \text{where } \sigma_{ij}^2 = Cov(Z_i, Z_j) \tag{5}$$

with a constraint $\quad 1 - \sum_{i=1}^{n}\lambda_i = 0$

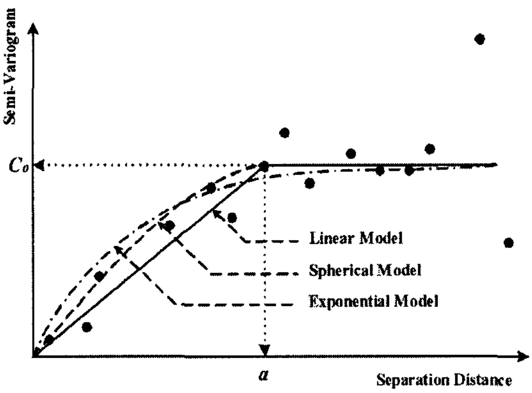

Fig. 1 Type of theoretical semi-variogram models

The Eq.(5) can be rewritten by Lagrange multiplier method.

$$L(\lambda_1,\lambda_2,\ldots,\lambda_n;\mu)=\sigma^2-2\sum_{i=1}^{n}\lambda_i\sigma_{oi}^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\sigma_{ij}^2+2\mu\left[1-\sum_{i=1}^{n}\lambda_i\right] \tag{6}$$

where $L(\lambda_1,\lambda_2,\cdots,\lambda_n;\mu)$ is Lagrange objective function, and μ is Lagrange multiplier. To minimize the objective function can be carried out by finding the partial derivatives with respect to λ_i and μ such that:

$$\frac{\partial L}{\partial \lambda_i}=-2\sigma_{ol}^2+2\sum_{i=1}^{n}\lambda_i\sigma_{il}^2-2\mu=0,\quad l=1,2,\ldots,n$$
$$\frac{\partial L}{\partial \mu}=2\left[1-\sum_{i=1}^{n}\lambda_i\right]=0 \tag{7}$$

The Eq.(7) can be rearranged by Eq.(8), and we finally obtain the matrix equation based on OK estimates where σ_{oi}^2 represents the estimation variance between the expected value $Z(x_o)$ at the unsampled location (x_o) and known values $Z(x_i)$ at the sampled location (x_o).

$$\sum_{i=1}^{n}\lambda_i\sigma_{il}^2-\mu=\sigma_{ol}^2,\quad l=1,2,\ldots,n$$
$$\sum_{i=1}^{n}\lambda_i=1 \tag{8}$$

3. A *Posteriori* Error Estimate for *p*-Refinement

It is inevitable to slightly modify the existing S.P.R. technique proposed by Z/Z(Zienkiewicz & Zhu, 1992) because the number of sampling Gauss points in each element is different according to non-uniform or selective *p*-distribution. Also, shape functions for a hierarchical finite element are different from those for conventional finite elements. The elemental error measure in the Z/Z approach is quantified by computing the strain energy contained in the difference between the discontinuous *p*-version finite element solution at the sampling Gauss point and the smoothed solution. The energy norm of the error in the displacement field proposed by Z/Z is given by:

$$\|e_r\| = \left(\int_{\Omega} \left(Z^* - Z^p\right)^T [D]^{-1} \left(Z^* - Z^p\right) d\Omega \right)^{1/2} \tag{9}$$

where

Z^* : smoothed stress field by projection

Z^P: computed stress at Gauss points by non-uniform p-refinement

$[D]$: constitutive matrix

Ω : mesh domain

Thus the error in terms of stresses may be computed either exactly using the exact stress field; $\hat{Z}$, or estimated using Z^*.

$$e_\sigma = \hat{Z} - Z^p \approx Z^* - Z^p \tag{10}$$

in which the smoothed continuous stress field, Z^*, is derived the discontinuous finite element stresses Z^P. The estimated stress at the unsampled points can be computed by linear combination of stresses at the sampling-points(Gauss points) such as :

$$Z^*(x_o, y_o) = \sum_{j=1}^{k} \lambda_j Z_j(x, y) \tag{11}$$

where k is number of Gauss points, and λ_j, $Z_j(x,y)$represent weight factors. Respectively. The energy norm of the displacement field itself, $\| \hat{r} \|$, may also be expressed in terms of stresses.

$$\|\hat{r}\| = \left(\int_{\Omega} \left(Z^p\right)^T [D]^{-1} Z^p \, d\Omega \right)^{1/2} \tag{12}$$

Thus the relative percentage error can be defined as:

$$\eta_\Omega = \left(\frac{\| e_r \|^2}{\| e_r \|^2 + \| \hat{r} \|^2} \right)^{1/2} \quad (\%) \tag{13}$$

In the presence of singularities, the asymptotic convergence behavior of the p-version of the finite element method permits a close estimate of the exact strain energy by extrapolation that is called the *limit value*, and hence we can predict the error in the energy norm on the finite element mesh employed. For a two-dimensional problem, under the assumption that the error in the energy norm has entered the asymptotic range where U_{ex} and U_{fe} are the strain energy (*i.e.* the degree of freedom is sufficiently high), the rate of convergence for the p-version of FEM can be derived by the inverse theorem as:

$$\left| U_{ex} - U_{fe} \right| \le k / N_p^{2\alpha} \tag{14}$$

where U_{ex} and U_{fe} are the exact strain energy estimated by the limit value and the approximate strain energy by FEM, α is the strength of singularity, N_p and k are the degrees of freedom for the polynomial order p and a constant which depends on the mesh, respectively. There are three unknowns U_{ex}, k, and α in Eq. (12). By performing three successive extension processes, p-2, p-1, and p, which are in the asymptotic range, we have three equations for computing the unknowns. Cancelling α and k in Eq. (14), the following extrapolation equation can be derived as:

$$\frac{Log\dfrac{U_{ex}^{L}-U_{p}}{U_{ex}^{L}-U_{p-1}}}{Log\dfrac{U_{ex}^{L}-U_{p-1}}{U_{ex}^{L}-U_{p-2}}}=\frac{Log\dfrac{N_{p-1}}{N_{p}}}{Log\dfrac{N_{p-2}}{N_{p-1}}} \tag{15}$$

where U_p, U_{p-1}, and U_{p-2} are the strain energies when the polynomial orders are p, $p-1$, and $p-2$, and U_{ex}^{L} represents the limit value in terms of estimated exact strain energy. N_p, N_{p-1}, and N_{p-2} are the number of degrees of freedom for each analysis. Thus the energy norm of the error in the displacement field by using limit value is given by:

$$\|e\|_E=\left[\frac{U_{ex}^{L}-U_{p}}{U_{ex}^{L}}\right]^{1/2} \tag{16}$$

Similar to the relative percentage error based on S.P.R. method, the new error indicator can be defined in Eq. (17) and also can be used to verify the performance of the modified S.P.R. method.

$$\zeta=\left[\frac{U_{ex}^{L}-U_{p}}{U_{ex}^{L}}\right]^{1/2} \quad (\%) \tag{17}$$

4. Numerical Example

The geometry of a centrally cracked panel(CCP) under simple tensile loading is shown in Fig. 2. Due to symmetry, a quarter of panel is modeled by the p-adaptive mesh refinement. A quarter of the panel with height $2h$, width $2b$, and crack length $2a$ is discretized into eight elements where $h/b=2$, $\sigma=1.0$, $E=2\times10^{6}$, and $\nu=0.3$. The final adaptive mesh is produced automatically by the computer program developed for this study as shown in Fig. 3 and Fig. 4. The LSM(least square method) and Ordinary Kriging(OK) are used to calculate the estimated exact solution Z^* (or smoothed stress field by projection), respectively. It is noted that the distribution of p-levels by LSM based adaptive mesh in the vicinity of crack tip is much higher than those by p-adaptive mesh associated with Ordinary Kriging technique.

However, we can obtain similar level of accuracy by OK based adaptive mesh that may be compared with not only LSM based adaptive mesh but also other empirical solutions by Irwin, *etc.* The non-dimensional stress intensity factors are plotted as a/bratio varies from 0.1 to 0.9. Also, the required number of iterations to determine final p-adaptive mesh are 9 for LSM based adaptive mesh and 4 for OK based adaptive mesh, respectively. The number of degree of freedoms(NDF) is 215 for LSM approach, on the other hand only 135 for OK technique. Thus it may be concluded that the solution of non-dimensional stress intensity factor is converged from NDF=135 shown in Table 2 and we can determine quickly the final adaptive mesh by OK technique comparing with LSM approach.

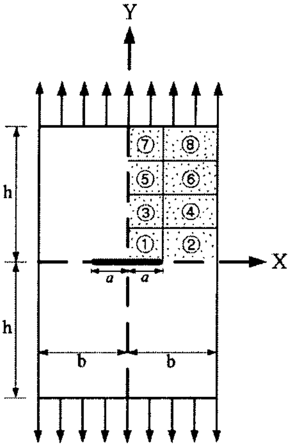

Fig. 2 Geometric configuration of a centrally cracked panel and finite element model.

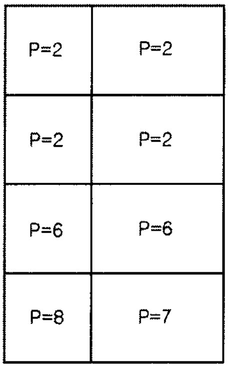

Fig. 3 Final adaptive mesh a/b=0.5 (LSM)

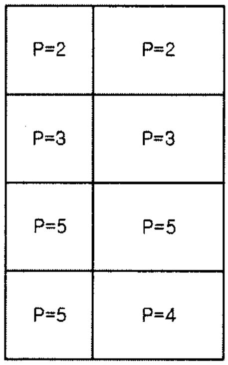

Fig. 4 Final adaptive mesh a/b=0.5 (OK)

Table 1 Non-dimensional stress intensity factors of CCP with respect to a/b ratio.

a/b	Irwin	Brown	Feddersen	Dixon	p-adaptive mesh(LSM)	p-adaptive mesh(OK)	$p = 8$ (all meshes)
0.1	1.004	1.011	1.006	1.005	0.950	0.931	0.977
0.2	1.017	1.026	1.025	1.0211	0.994	0.984	1.001
0.3	1.040	1.054	1.059	1.048	1.033	1.007	1.035
0.4	1.075	1.103	1.112	1.091	1.086	1.054	1.085
0.5	1.128	1.183	1.189	1.155	1.162	1.123	1.158
0.6	1.208	1.303	1.304	1.250	1.274	1.251	1.267
0.7	1.336	1.473	1.484	1.400	1.448	1.420	1.437
0.8	1.565	1.670	1.799	1.667	1.748	1.700	1.729
0.9	2.114	1.994	2.528	2.294	2.420	2.331	2.384

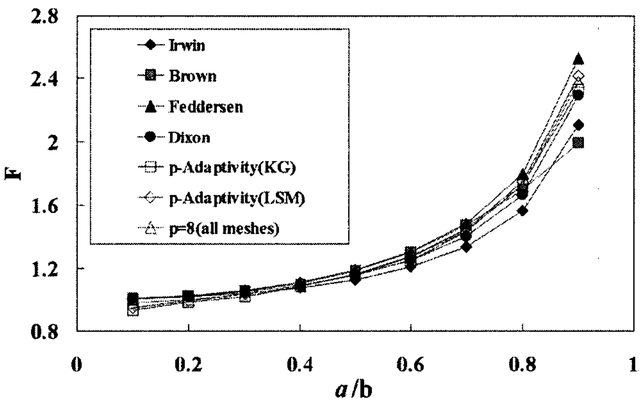

Fig. 4 Comparison of non-dimensional stress intensity factors as a/b ratio varies from 0.1 to 0.9.

Table 2 The relative percentage error based on S.P.R. method when a/b=0.5.

No. of Iteration	Least Square Method		Ordinary Kriging	
	NDF	η_Ω(%)	NDF	η_Ω(%)
1	23	10.77	23	11.99
2	49	13.34	49	9.58
3	66	11.35	79	5.66
4	83	10.91	104	4.06
5	113	10.73	135	2.73
6	140	10.92	-	-
7	168	10.39	-	-
8	203	10.10	-	-
9	215	9.92	-	-

As sown in Table 3 and Table 4, the required numbers of degree of freedom to produce the LSM based p-adaptive mesh are between ten and eleven iterations, on the other hand, the OK based p-adaptive mesh requires only five and six iterations to determine the final adaptive mesh. This means that the accuracy of non-dimensional stress intensity factors is virtually unchanged after the number of iteration is six in case of the LSM based approach.

To clarify the relative errors in Table 2 Table 5, the results by the conventional S.P.R. method are compared with those by the limit value approach. As mentioned before, the first method is based on the errors of energy norm between stresses calculated at Gauss points by FEM analysis and those by the smoothed stress field(estimated exact stress function) using projection technique such as LSM or OK method. However, the second method is associated with same stresses calculated at Gauss points by FEM analysis and the exact stress function by limit value approach proposed in this study. When the relative percentage error(η_Ω(%)) by LSM when NDF=140 is 10.92%, but that by OK method yields 2.73% when NDF=135 shown in Table 2. The relative percentage errors(ζ(%)) based on Limit Value method when same NDF and a/b are used are 16.04% and 14.56%, respectively. It may be also noted that the relative percentage error based on Limit Value method is gradually decreased as NDF and number of iteration are increased. Thus it can be concluded that the error indicator by the conventional S.P.R. method is merely used to check whether the mesh should be refined more or the polynomial order of element should be increased.

Table 3 NDF with respect of a/b ratio by LSM based p-adaptive mesh.

No. of Iteration	a/b								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	23	23	23	23	23	23	23	23	23
2	26	34	34	49	49	54	62	62	62
3	43	55	55	66	66	79	79	73	79
4	62	73	78	78	83	104	104	96	107
5	82	86	94	107	113	123	130	130	140
6	91	119	127	116	140	150	159	159	169
7	115	138	146	145	168	174	194	170	191
8	137	169	172	168	203	196	213	194	217
9	177	194	182	185	215	217	226	231	223
10	194	198	-	191	-	223	231	-	-
11	210	-	-	-	-	-	-	-	-

Table 4 NDF with respect of a/b ratio by OK based p-adaptive mesh.

No. of Iteration	a/b								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	23	23	23	23	23	23	23	23	23
2	28	35	44	49	49	54	54	54	62
3	40	52	61	67	79	74	87	65	87
4	66	66	75	84	104	103	97	78	130
5	86	95	102	91	135	110	104	98	137
6	–	–	118	111	–	141	113	107	146
7	–	–	–	–	–	–	–	–	–
8	–	–	–	–	–	–	–	–	–

Table 5 The relative percentage error based on Limit Value method when a/b=0.5.

No. of Iteration	Least Square Method		Ordinary Kriging	
	NDF	$\zeta(\%)$	NDF	$\zeta(\%)$
1	23	31.83	23	30.75
2	49	22.29	49	20.65
3	66	20.94	79	19.17
4	83	18.37	104	17.06
5	113	16.93	135	14.56
6	140	16.04	–	–
7	168	15.44	–	–
8	203	13.79	–	–
9	215	13.78	–	–

5. Conclusions

The new p-adaptive finite element model with Ordinary Kriging technique is proposed in this study. This approach shows superior performance to the existing SPR method by Z/Z using LSM(least square method) to estimated smoothed stress field(estimated exact solution by FEM) by projection. This proposed model is very suitable for stress singularity problems like fracture mechanics since the high weight factor is used near the crack tip to interpolate the stress values at the Gauss points that is calculated by variogram model and Kriging interpolation technique.

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