Fuzzy Weakly r-Semicontinuous Maps
퍼지 weakly r-반연속 사상

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Abstract

By generalizing the definition of B. S. Zhong, we introduce the concept of fuzzy weakly r-semicontinuous maps in fuzzy topology of Chattopadhyay. Then the concept introduced by B. S. Zhong become special case of our definition. Also, we show that fuzzy weakly r-semicontinuity and fuzzy weakly r-continuity are independent notions.

Keywords : fuzzy weakly r-semicontinuous

1. Introduction

Chang [1] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. B. S. Zhong [7] introduced the concept of fuzzy weakly semicontinuous maps in Chang's fuzzy topology. Chattopadhyay and his colleagues [2, 3] introduced another definition of fuzzy topology as a generalization of Chang's fuzzy topology. By generalizing the definition of B. S. Zhong, we introduce the concept of fuzzy weakly r-semicontinuous maps in fuzzy topology of Chattopadhyay. Then the concept introduced by B. S. Zhong become special case of our definition. Also, we show that fuzzy weakly r-semicontinuity and fuzzy weakly r-continuity are independent notions.

2. Preliminaries

We will denote the unit interval $[0, 1]$ of the real line by I and $I_0 = (0, 1]$. A member μ of I^X is called a *fuzzy set* in X. For any $\mu \in I^X$, μ^c denotes the complement $1 - \mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.

(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.

(3) If $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.*

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

For each $\alpha \in (0, 1]$, a *fuzzy point* x_α is a fuzzy set in X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

In this case, x and α are called the *support* and the *value* of x_α, respectively. A fuzzy point x_α is said to *belong* to a fuzzy set μ in X, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$.

Definition 2.1. ([5]) Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-open* if $\mathcal{T}(\mu) \geq r$,

(2) *fuzzy r-closed* if $\mathcal{T}(\mu^c) \geq r$.

Definition 2.2. ([5, 6]) Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$,

(3) *fuzzy r-regular open* if $\mathrm{int}(\mathrm{cl}(\mu, r), r) = \mu$,

(4) *fuzzy r-regular closed* if $\mathrm{cl}(\mathrm{int}(\mu, r), r) = \mu$.

Theorem 2.3. ([5]) Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:

(1) μ is a fuzzy r-semiopen set.

(2) μ^c is a fuzzy r-semiclosed set.

(3) $\mathrm{cl}(\mathrm{int}(\mu, r), r) \geq \mu$.

(4) $\mathrm{int}(\mathrm{cl}(\mu^c, r), r) \leq \mu^c$.

Definition 2.4. ([5]) Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-semiclosure* is defined by

$$\mathrm{scl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\},$$

and the *fuzzy r-semiinterior* is defined by

$$\mathrm{sint}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}.$$

Obviously $\mathrm{scl}(\mu, r)$ is the smallest fuzzy r-semiclosed set which contains μ and $\mathrm{sint}(\mu, r)$ is the greatest fuzzy r-semiopen set which is contained in μ. Also, $\mathrm{scl}(\mu, r) = \mu$ for any fuzzy r-semiclosed set μ and $\mathrm{sint}(\mu, r) = \mu$ for any fuzzy r-semiopen set μ. Moreover, we have

$$\mathrm{int}(\mu, r) \leq \mathrm{sint}(\mu, r) \leq \mu \leq \mathrm{scl}(\mu, r) \leq \mathrm{cl}(\mu, r).$$

Also, we have the following results :

(1) $\mathrm{scl}(\tilde{0}, r) = \tilde{0}$, $\mathrm{scl}(\tilde{1}, r) = \tilde{1}$, $\mathrm{sint}(\tilde{0}, r) = \tilde{0}$, $\mathrm{sint}(\tilde{1}, r) = \tilde{1}$.

(2) $\mathrm{scl}(\mu, r) \geq \mu$, $\mathrm{sint}(\mu, r) \leq \mu$.

(3) $\mathrm{scl}(\mu \vee \rho, r) \geq \mathrm{scl}(\mu, r) \vee \mathrm{scl}(\rho, r)$, $\mathrm{sint}(\mu \wedge \rho, r) \leq \mathrm{sint}(\mu, r) \wedge \mathrm{sint}(\rho, r)$.

(4) $\mathrm{scl}(\mathrm{scl}(\mu, r), r) = \mathrm{scl}(\mu, r)$, $\mathrm{sint}(\mathrm{sint}(\mu, r), r) = \mathrm{sint}(\mu, r)$.

Theorem 2.5. ([4]) For a fuzzy set μ in a fuzzy topological space X and $r \in I_0$, we have :

(1) $\mathrm{sint}(\mu, r)^c = \mathrm{scl}(\mu^c, r)$.

(2) $\mathrm{scl}(\mu, r)^c = \mathrm{sint}(\mu^c, r)$.

Definition 2.6. ([4, 5, 6]) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set in X for each fuzzy r-open set μ in Y,

(2) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set in X for each fuzzy r-open set μ in Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-semiclosed set in X for each fuzzy r-closed set μ in Y,

(3) a *fuzzy almost r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set in X for each fuzzy r-regular open set μ in Y,

(4) a *fuzzy weakly r-continuous* map if $f^{-1}(\mu) \leq \mathrm{int}(f^{-1}(\mathrm{cl}(\mu, r)), r)$ for each fuzzy r-open set μ in Y.

(5) a *fuzzy r-irresolute* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set in X for each fuzzy r-semiopen set μ in Y.

3. Fuzzy weakly r-semicontinuous maps

We define the notion of fuzzy weakly r-semicontinuous maps, and investigate some of their properties.

Definition 3.1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is called a fuzzy weakly r-semicontinuous map if $f^{-1}(\mu) \leq \mathrm{sint}(f^{-1}(\mathrm{scl}(\mu, r)), r)$ for each fuzzy r-open set μ in Y.

Remark 3.2. It is obvious that a fuzzy r-semicontinuous map is also a fuzzy weakly r-semicontinuous map for each $r \in I_0$. But the converse does not hold as in the following example.

Example 3.3. Let $X = \{x, y, z\}$ and μ_1 and μ_2 be fuzzy sets in X defined as

$$\mu_1(x) = \frac{1}{3},\ \mu_1(y) = \frac{1}{3},\ \mu_1(z) = \frac{1}{2};$$

and

$$\mu_2(x) = \frac{1}{2},\ \mu_2(y) = \frac{1}{2},\ \mu_2(z) = \frac{1}{2}.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1,\ \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X. Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ for each $x \in X$. Thus f is fuzzy weakly $\frac{1}{2}$-semicontinuous map. But $f^{-1}(\mu_1) = \mu_1$ is not fuzzy $\frac{1}{2}$-semiopen in $(X, \mathcal{T}_1)$ and hence f is not a fuzzy $\frac{1}{2}$-semicontinuous map.

Theorem 3.4. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a fuzzy almost r-continuous map. Then f is also a fuzzy weakly r-semicontinuous map.

That the converse of Theorem 3.4 need not be true is shown by the following example.

Example 3.5. Let $X = I$ and μ_1 and μ_2 be fuzzy sets in X defined as

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \le x \le 1; \end{cases}$$

and

$$\mu_2(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{4}, \\ -4x + 2 & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1,\ \mu_2,\ \mu_1 \vee \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a fuzzy topology on X. Let $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ be defined by $f(x) = \frac{1}{2}x$. It is easy to see that $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}, f^{-1}(\mu_1) = \tilde{0}$ and $f^{-1}(\mu_2) = f^{-1}(\mu_1 \vee \mu_2) = \mu_1^c$. Since $\mathrm{cl}(\mu_2, \frac{1}{2}) = \mu_1^c$, μ_1^c is a fuzzy $\frac{1}{2}$-semiopen set and thus f is a fuzzy $\frac{1}{2}$-semicontinuous map. Hence f is a fuzzy weakly $\frac{1}{2}$-semicontinuous map. Note that $\mathrm{int}(\mathrm{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \mu_2$. Thus μ_2 is a fuzzy $\frac{1}{2}$-regular open set in Y. But $f^{-1}(\mu_2) = \mu_1^c$ is not fuzzy $\frac{1}{2}$-open. Hence f is not a fuzzy almost $\frac{1}{2}$-continuous map.

Remark 3.6. That a fuzzy weakly r-semicontinuous map need not be a fuzzy weakly r-continuous map and a fuzzy weakly r-continuous map need not be a fuzzy weakly r-semicontinuous map is shown in the following examples.

Example 3.7. A fuzzy weakly r-semicontinuous map need not be a fuzzy weakly r-continuous map.

Let $X = \{x, y, z\}$ and μ_1, μ_2 and μ_3 be fuzzy sets in X defined as

$$\mu_1(x) = \frac{3}{10},\ \mu_1(y) = \frac{1}{10},\ \mu_1(z) = \frac{1}{10};$$

$$\mu_2(x) = \frac{1}{2},\ \mu_2(y) = \frac{1}{2},\ \mu_2(z) = \frac{1}{2};$$

and

$$\mu_3(x) = \frac{1}{5},\ \mu_3(y) = \frac{1}{10},\ \mu_3(z) = 0.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1,\ \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X. Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ for each $x \in X$. Then f is a fuzzy weakly $\frac{1}{2}$-semicontinuous map. f is not a fuzzy weakly $\frac{1}{2}$-continuous map.

Example 3.8. A fuzzy weakly r-continuous map need not be a fuzzy weakly r-semicontinuous map.

From the above two examples, we have the following result.

Theorem 3.9. Fuzzy weakly r-semicontinuous maps and fuzzy weakly r-continuous maps are independent notions.

From the above definitions and theorems one may easily verify the following implications.

Theorem 3.10. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then the following statements are equivalent:

(1) f is a fuzzy weakly r-semicontinuous map.

(2) $\mathrm{scl}(f^{-1}(\mathrm{sint}(\mu, r)), r) \leq f^{-1}(\mu)$ for each fuzzy r-closed set μ in Y.

(3) $f^{-1}(\mathrm{int}(\rho, r)) \leq \mathrm{sint}(f^{-1}(\mathrm{scl}(\rho, r)), r)$ for each fuzzy set ρ in Y.

(4) $\mathrm{scl}(f^{-1}(\mathrm{sint}(\rho, r)), r) \leq f^{-1}(\mathrm{cl}(\rho, r))$ for each fuzzy set ρ in Y.

Definition 3.11. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is said to be *fuzzy weakly r-semicontinuous at a fuzzy point x_α* in X if for each fuzzy r-open set μ in Y and $f(x_\alpha) \leq \mu$, there exists a fuzzy r-semiopen set ρ in X such that $x_\alpha \in \rho$ and $f(\rho) \leq \mathrm{scl}(\mu, r)$.

Theorem 3.12. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is fuzzy weakly r-semicontinuous if and only if f is fuzzy weakly r-semicontinuous for each fuzzy point x_α in X.

Let $(X, \mathcal{T})$ be a fuzzy topological space. For an r-cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X, \mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Recall [2] that a fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu = T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.13. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is fuzzy weakly r-semicontinuous if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy weakly semicontinuous.

Theorem 3.14. Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to a Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy weakly semicontinuous if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy weakly r-semicontinuous.

Remark 3.15. By the above two theorems, we know that the concept of a fuzzy weakly r-semicontinuous map is a generalization of the concept of a fuzzy weakly semicontinuous map.

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