계수려진을 받는 단순지지 보의 비선형 진동특성

Nonlinear Analysis of Simply supported Elastic Beams under Parametric Excitation

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Key Words : Cosserat theory(Cosserat 이론), Parametric excitation(계수려진), Nonlinear vibration(비선형진동).

ABSTRACT

This paper presents the nonlinear characteristics of the parametric resonance of a simply supported beam which is inextensible beam. For the beam model, the order-three expanded equation of motion has been determined in a form amenable to a perturbation treatment. The equation of motion is derived by a special Cosserat theory. The method of multiple scales is used to determine the equations that describe to the first-order modulation of the amplitude of simply supported beam. The stability and the bifurcation points of the system are investigated applying the frequency response function.

1. Introduction

Research on the dynamics of large-amplitude of the beam structures is important for engineering and its applications. The large-amplitude motions are excited around resonances with finite displacements and rotations whereas the strains often remain small. The most comprehensive theory today available to describe overall motions of rods is the special Cosserat theory of rods [1]. The beam is mathematically conceived as a one-dimensional continuum with a local rigid structure. Because of the postulated local rigidity, the sections cannot undergo distortion and warping deformations; therefore, the theory is mainly restricted to beams with closed cross sections. Although many attempts have been made along this line, it is difficult to solve Cossserat's basic equations rigorously because of its high generality and nonlinearity [2]. Parametric resonance, which is characterized by the harmonic variation of coefficients of differential equations, is well known and has been studied by numerous authors [3,4]. The response of linear and nonlinear system subjected to multi-frequency parametric excitation have been investigated [5,6].

Yabuno *et al.*[7] studied the nonlinear analysis of a parametrically excited cantilever beam. In particular, the effect of the tip mass on the nonlinear characteristics of the frequency-response is theoretically and experimentally presented.

In the present research, we theoretically analyzed the nonlinear characteristics of the simply supported beam under the parametric resonance for the first-order modulation. The equation of motion is derived by a special Cosserat theory and analyzed by the method of multiple scales.

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2. Analytical model and equation of motion

We consider a simply supported beam subjected to a periodic excitation as shown in Fig. 1. The periodic excitation is $\xi = a_e \cos\Omega t$, where a_e, Ω and t are the excitation amplitude, frequency and the time, respectively. The notation employed in this analysis is as follows: ρA is mass of the beam per unit length; EI is the bending stiffness coefficient; m and l are the tip mass and the length of the beam, respectively.

Denoting with $\mathbf{e}_j \ (j=1,2,3)$ the orthonormal vectors of a fixed inertial reference frame such that $\mathbf{e}_1$ is parallel to the beam base curve, the position of a material point along the beam axis is represented by the vector $\mathbf{X}(x,t) = x\mathbf{e}_1$, where x indicates the coordinate along the straight undeformed beam axis with the origin O fixed at the left end of the beam. Thus, the material section at P is specified by the pair of orthonormal vectors $\mathbf{a}_j \ (j=1,2,3)$. We set $\mathbf{a}_1 = \mathbf{a}_2 \times \mathbf{a}_3$ so that $\{\mathbf{a}_j\}$ are a right-handed orthonormal basis for the 3-dimensional Euclidean space.

2.1 Prestressed equilibrium states

The displacement in the x-axis is assumed to be defined by the position vector $\mathbf{x}(x,t) = \mathbf{X}(x) + \mathbf{u}(x,t)$ with $\mathbf{u} = U\mathbf{a}_1 + v\mathbf{a}_2$ denoting the displacement vector from P to P_O and by the pair of orthonormal directors $\mathbf{d}_j$ where

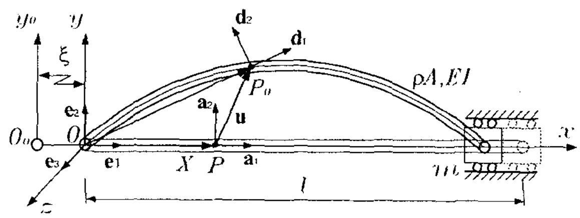

Fig. 1 Analytical model of the beam

$U = \xi + u$. The directors $\mathbf{d}_1$ and $\mathbf{d}_2$ are obtained from $\mathbf{a}_j$ via a finite rotation about the $\mathbf{d}_3$-axis, described by the proper orthogonal rotation tensor $\mathbf{R}(x,t)$, restricted to the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, as

$$\mathbf{d}_j = \mathbf{R}\mathbf{a}_j, \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

Therefore $\mathbf{a}_j$ and $\mathbf{d}_j$ are as follows, respectively:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{d}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{2}$$

Consequently, the only nontrivial strains are the axial strain and the bending curvature, the non-zero component of the curvature tensor $\mathbf{K}$, and are expressed as

$$\varepsilon_s = \sqrt{(1+u')^2 + v'^2} - 1, \qquad k = \theta' = \frac{v'' + u'v'' - u''v'}{(1+\varepsilon_s)^2} \tag{3}$$

2.2 Equilibrium equation

To obtain the equilibrium equations using the components of the contact force, body force and couple which are expressed using the current set of directors $\mathbf{d}_j$ as

$$\mathbf{n}(x,t) = N\mathbf{d}_1 + H\mathbf{d}_2 \tag{4}$$

$$\mathbf{b}(x,t) = b_1\mathbf{d}_1 + b_2\mathbf{d}_2 \tag{5}$$

$$\mathbf{m}(x,t) = M\mathbf{d}_3 \tag{6}$$

where N and H denote the axial load and shear force, respectively. b_1 and b_2 indicate the body force per unit reference length along the $\mathbf{d}_1$ and $\mathbf{d}_2$ directions, respectively, and M is the bending moment. The equilibrium equations, requiring the balance of linear momentum and moment of momentum, are

$$\mathbf{n}'(x,t) + \mathbf{b}(x,t) = \mathbf{0} \tag{7}$$

$$\mathbf{m}'(x,t) + \mathbf{x}'(x,t) \times \mathbf{n}(x,t) + \mathbf{c}(x,t) = \mathbf{0} \tag{8}$$

where [$\times$] denotes the vector product. The mechanical boundary conditions are

$$\begin{aligned} x = 0:&\ \mathbf{n}(x,t) = \mathbf{n}(0,t), \quad \mathbf{m}(x,t) = \mathbf{m}(0,t) \\ x = l:&\ \mathbf{n}(x,t) = \mathbf{n}(l,t), \quad \mathbf{m}(x,t) = \mathbf{m}(l,t) \end{aligned} \tag{9}$$

The equilibrium equations, after filtering out the shear force H, become

$$N' + \left(\frac{k}{1+\varepsilon_s}\right)M' + \left(\frac{k}{1+\varepsilon_s}\right)c + b_1 = 0 \tag{10}$$

$$\left(\frac{M'}{1+\varepsilon_s}\right)' + \left(\frac{c}{1+\varepsilon_s}\right)' - kN - b_2 = 0 \tag{11}$$

By virtue of D'Alembert's principle, the body forces and the corresponding couple are expressed as

$$\begin{aligned} \mathbf{b}(x,t) &= -\rho A\left(\ddot{U}\,\mathbf{a}_1 + \ddot{v}\,\mathbf{a}_2\right) \\ &= -\rho A\left[\left(\ddot{U}\cos\theta + \ddot{v}\sin\theta\right)\mathbf{d}_1 + \left(-\ddot{U}\sin\theta + \ddot{v}\cos\theta\right)\mathbf{d}_2\right] \end{aligned} \tag{12}$$

$$\mathbf{c}(x,t) = -\rho I\ddot{\theta}\mathbf{a}_3 \tag{13}$$

where [$\cdot$] indicates the differentiation with respect to time t. To inertially uncouple the equations of motion, we project them into the $(\mathbf{a}_1, \mathbf{a}_2)$ basis and obtain

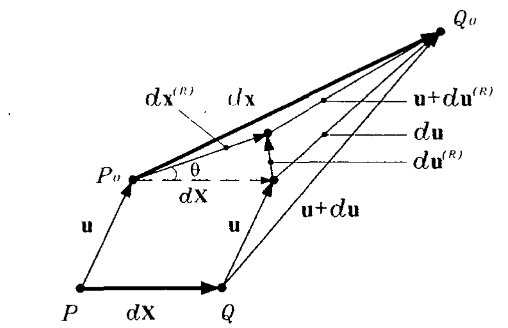

Fig. 2 Strain of the beam element

$$\begin{aligned} &\rho A\left(\ddot{\xi} + \ddot{u}\right) - \left[N' + \frac{k}{1+\varepsilon_s}M' + \frac{k}{1+\varepsilon_s}\left(-\rho I\ddot{\theta}\right)\right]\cos\theta \\ &\quad - \left[\left(\frac{M'}{1+\varepsilon_s}\right)' - kN + \left(\frac{-\rho I\ddot{\theta}}{1+\varepsilon_s}\right)'\right]\sin\theta = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} &\rho A\ddot{v} - \left[N' + \frac{k}{1+\varepsilon_s}M' + \frac{k}{1+\varepsilon_s}\left(-\rho I\ddot{\theta}\right)\right]\sin\theta \\ &\quad + \left[\left(\frac{M'}{1+\varepsilon_s}\right)' - kN + \left(\frac{-\rho I\ddot{\theta}}{1+\varepsilon_s}\right)'\right]\cos\theta = 0 \end{aligned} \tag{15}$$

The boundary conditions are

$$u(0,t) = 0, \quad v(0,t) = v(l,t) = 0, \quad M(0,t) = M(l,t) = 0 \tag{16}$$

$$N\cos\theta + \frac{M'}{1+\varepsilon_s}\sin\theta = -m\left(\ddot{\xi} + \ddot{u}\right) \quad at \quad x = l \tag{17}$$

where m is the tip mass. From a constitutive point of view, because the axial strains are assumed small and the curvature is finite but moderately large, linear constitutive equations are assumed in the standard uncoupled form

$$N(x,t) = EA(x)\varepsilon_s(x,t), \qquad M(x,t) = EI(x)k(x,t) \tag{18}$$

where E stands for Young's modulus, A and I denote the area and moment of inertia of the cross section of the beam, respectively.

2.3 Equation of motion for inextensible beam

When the beam is axially unrestrained to enforce vanishing of the axis elongation, $\varepsilon_s = 0$, which leads to

$$(1+u')^2 + v'^2 = 1 \tag{19}$$

On account of $\varepsilon_s = 0$, the exact bending curvature, $k = \theta'$, and its third-order expansion become

$$k = v'' + u'v'' - u''v' \approx v'' + \frac{1}{2}v'^2 v'' \tag{20}$$

Moreover, $\sin\theta = v'$ and $\cos\theta \approx 1 + u' \approx 1 - 1/2(v')^2$.

The equilibrium equation is Eqs. (10), (11) with $\varepsilon = 0$. Therefore, the axial force along the beam can be obtained by integrating Eq. (10) with $\varepsilon = 0$ as follows:

$$N(x,t) = N(l,t) - \int_l^x kM'\,dx - \int_l^x b_1\,dx \tag{21}$$

Using the mechanical boundary condition in the axial direction at $x = l$ given by Eq. (3), the axial force and the equilibrium equation in the transverse direction become

$$N(x,t) = -\int_l^x kM'\,dx - \int_l^x b_1\,dx - M'(l,t)\tan\theta(l,t) - \frac{m\ddot{U}(l,t)}{\cos\theta(l,t)} \tag{22}$$

$$M'' + k\left[M'\tan\theta + m\ddot{U}\sec\theta\right]\Big|_{x=l} + k\int_l^x kM'\,dx + \int_l^x b_1\,dx - b_2 = 0 \tag{23}$$

Incorporating the longitudinal motion into the inertial forces, expanding the resulting equation and retaining terms up to third order yield the equation of motion.

3. Nonlinear analysis

To obtain the dimensionless equation of motion, the following dimensionless variables and parameters are introduced:

$$t^* = \frac{t}{T},\quad x^* = \frac{x}{l},\quad v^* = \frac{v}{l},\quad a_e^* = \frac{a_e}{l},\quad m^* = \frac{m}{\rho A l} \tag{24}$$

where $T^2 = \rho A l^4 / EI$. The resulting dimensionless equation of motion by taking into account a viscous damping effect, dropping the star for sake of notational simplicity and neglecting the rotary term along with the distributed couples, is

$$\begin{aligned}
&\ddot{v} + v'\int_0^x\left(\dot{v}'^2 + v'\ddot{v}'\right)dx - \frac{1}{2}\ddot{v}v'^2 + 2\mu\dot{v} + v'''' + v''^3 + 3v'v''v''' \\
&+\frac{1}{2}v'^2v'''' + v''\left[v'v'''\right]\Big|_{x=1} + v''\int_1^x v''v'''dx - mv''\int_0^1\left(\dot{v}'^2 + v'\ddot{v}'\right)dx \\
&+ v''\int_1^x\left[-v'\ddot{v} - \frac{a_e(\Omega v')^2}{2}\cos\Omega t + \int_0^x\left(\dot{v}'^2 + v'\ddot{v}'\right)dx\right]dx \\
&+ a_e\Omega^2\left[v' - (1-x)v''\right]\cos\Omega t - ma_e\Omega^2 v''\left(1 + \frac{1}{2}v'^2\right)\cos\Omega t = 0
\end{aligned} \tag{25}$$

where μ is the damping ratio in the transverse direction. The boundary conditions are

$$v(0) = v''(0) = v(1) = v''(1) = 0 \tag{26}$$

For the case of principal parametric resonance of the first mode of the beam, we put

$$\Omega = 2\omega_1 + \sigma,\quad \sigma = \varepsilon\hat{\sigma} \tag{27}$$

where ε is a small parameter $(|\varepsilon| \ll 1)$ of book-keeping device and σ is a detuning parameter and [^] denotes $O(1)$. We also set a_e and μ, as $a_e = \varepsilon\hat{a}_e$ and $\mu = \varepsilon\hat{\mu}$, respectively. By the method of multiple scales [8], we analyze the equation of motion Eq. (25). The uniform expansions of the solution of Eq. (25) is sought in the form

$$v = \varepsilon^{1/2}v_1 + \varepsilon^{3/2}v_2 + \cdots \tag{28}$$

Multiple time scales are introduced as follows:

$$t_0 = t,\ t_2 = \varepsilon t \tag{29}$$

We substitute Eq. (29) into the system of equation of motion Eq. (25) and boundary conditions, use the independence of the time scales, equate coefficients of like powers of ε, and obtain the following:

$$O\left(\varepsilon^{1/2}\right): \quad D_0^2v_1 + v_1'''' = 0 \tag{30}$$

$O\left(\varepsilon^{3/2}\right)$:

$$\begin{aligned}
D_0^2v_3 + v_3'''' = &-2D_0D_2v_1 - v_1'\int_0^x\left(D_0v_1'^2 + v_1'D_0v_1'\right)dx + \frac{1}{2}v_1'^2D_0^2v_1 \\
&- 2\hat{\mu}D_0v - v_1''^3 - 3v'v''v''' - \frac{1}{2}v_1'^2v_1'''' - v_1''\left[v_1'^2v_1'''\right]\Big|_{x=1} \\
&- v_1''\int_1^x\left[-v_1'D_0^2v_1 + \int_0^x\left(D_0v_1'^2 + v_1'D_0v_1'\right)dx\right]dx \\
&+ mv_1''\int_0^1\left(D_0v_1'^2 + v_1'D_0v_1'\right)dx - v_1''\int_1^x v_1''v_1'''dx \\
&- 4\hat{a}_e\omega_1^2\left[v_1' + (x-m-1)v_1''\right]\cos\Omega t
\end{aligned} \tag{31}$$

where $D_n = \partial/\partial t_n$. The boundary conditions are

$$\begin{aligned}
&v_1(0) = v_1''(0) = v_1(1) = v_1''(1) = 0, \\
&v_3(0) = v_3''(0) = v_3(1) = v_3''(1) = 0
\end{aligned} \tag{32}$$

With this approach it turns out to be convenient to write the solution of Eq. (31) in the complex form

$$v_1 = \left\{A(t_2)e^{i\omega_1 t_0} + cc\right\}\phi_1(x) \tag{33}$$

where cc denotes the complex conjugate of the preceding term. We can obtain the mode shape as $\phi_1(x) = \sqrt{2}\sin\pi x$. By considering the boundary condition Eq. (32) and substituting Eq. (32) into Eq. (31), we obtain the solvability condition as follows:

$$2i\omega_1\left(D_2A + \hat{\mu}A\right) + \beta_1\Gamma_1A^2\bar{A} + 2\hat{a}_e\beta_1\beta_2\omega_1^2e^{i\hat{\sigma}t_2}\bar{A} = 0 \tag{34}$$

Substituting the form $A = B(t_2)e^{i\hat{\sigma}t_2/2}$ and the polar form $B = \frac{1}{2}\hat{a}e^{i\beta(t_2)}$ into Eq. (34) we get

$$\begin{aligned}
&\left(\frac{da}{dt} + ia\frac{d\beta}{dt}\right) + \frac{\sigma}{2}a + \mu a - \frac{i}{8\omega_1}a^3\beta_1\Gamma_1 \\
&\quad - a\beta_1\beta_2a_e\omega_1\left(\cos 2\beta + i\sin 2\beta\right) = 0
\end{aligned} \tag{35}$$

Separating real and imaginary parts in Eq. (35) yields

$$\frac{da}{dt} = -\mu a + a\beta_1\beta_2a_e\omega_1\cos 2\beta \tag{36}$$

$$a\frac{d\beta}{dt} = -\frac{\sigma}{2}a + \frac{i}{8\omega_1}a^3\beta_1\Gamma_1 + a\beta_1\beta_2a_e\omega_1\sin 2\beta \tag{37}$$

Then, the first-order expansion of the solution of Eq. (25) is given by

$$v = a\cos\left(\frac{\Omega}{2}t + \beta\right)\phi_1(x) + cc + O(\varepsilon^{3/2}) \tag{38}$$

where a and β are defined by Eqs. (36) and (37). Further solving for the fixed points of the real-valued modulation equations resulting, the following frequency -response equation is obtained:

$$a = \left[\frac{8\omega_1}{\beta_1\Gamma_1}\left(\frac{\sigma}{2} \pm \sqrt{\left(a_e\beta_1\beta_2\omega_1\right)^2 - \mu^2}\right)\right]^{1/2} \tag{39}$$

Fig. 3 shows the stability of the trivial solution for $\mu = 0.04$ and three different values of the tip mass, namely $m = 2.8$, $m = 4.2$ and $m = 7.4$. The hatched re-

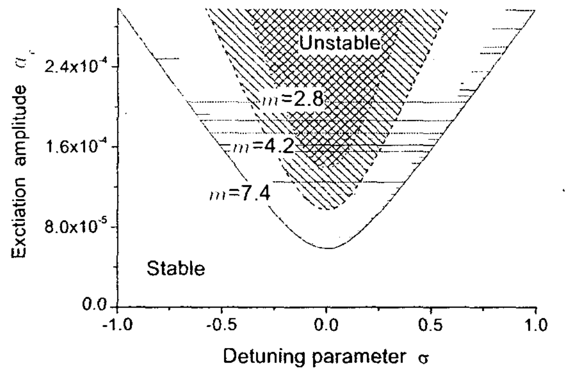

Fig. 3 Stability boundaries of the principal parametric resonance ($\mu = 0.04$)

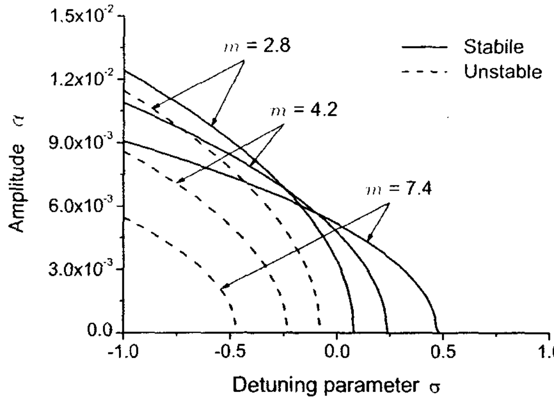

Fig. 4 Frequency-response curve of the principle parametric resonance ($a_e = 3.33\times10^{-4}$)

gion indicates the region where the trivial solution is unstable. When the excitation amplitude a_e is constant, as the tip mass is increased, the unstable region of the trivial solution is increased. This is likely due to the fact that the force generated by the tip mass having a fixed direction parallel to $\mathbf{a}_1$ produces a destabilizing effect when it is compressive due to the introduced negative stiffness.

Fig. 4 shows the influence of the tip mass on the frequency-response of the inextensible beam when the damping ratio is $\mu = 0.04$ and the dimensionless excitation amplitude is $a_e = 3.33\times10^{-4}$. The case $m = 7.4$ corresponds to the parameter values in the subsequent experiment. In Fig. 4, the solid (dashed) line shows the stable (unstable) branches. As the tip mass is increased, the frequency-response curve is more bent to the left. The increase of the tip mass makes the nonlinearity of the lowest mode more softening, while in the case of a parametrically excited cantilever beam, increasing the tip mass makes the mode hardening

4. Conclusions

This paper presents the nonlinear characteristics of the parametric resonance of simply supported elastic beams. The beam model, incorporating the inextensibility and unshearability constraints, describes bending motions only; hence, it is suitable for beams that are either axially unrestrained or weakly restrained. The equation of motion has been obtained employing the special Cosserat theory and its third-order perturbation has been determined in a form amenable to an asymptotic treatment. Via the nonlinear analysis based on the asymptotic solutions, the effects of the tip mass on the unstable region of the trivial response and the nonlinear characteristics have been investigated for a parametrically excited simply supported beam. It is shown that the increase of the tip mass produces a more pronounced softening effect, whereas increasing the tip mass in parametrically excited cantilever beams produces a hardening effect.

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