A Revisit to:

D'Alembert-Lagrange's Principal Balance Equations, Their Origin and Applications

(D'Alembert-Lagrange

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Principal Figures for Today's Lecture

D'Alembert's Principle was reported to Académie des Sciences in 1742.

Lagrange came to Paris from Turin in 1787 and published his book in 1788.

Traité de Dynamique (1758)

Méchanique Analytique (1788)

Developments in the Formulation of Dynamical Systems

- It began with a single, free and rigid body with a point mass;
- Then, a rigid body with the rotational inertia properties;
- A rigid-link with joints;
- Lumped mass-spring models
- Continuum (flexible) models
- FEM/Rigid models with joints and constraints;
- And, the race for complex models is on!

Theme of Today's Lecture

As a result of the race for ever more complex modeling and analysis, often physical insight with simplicity in modeling is lost or abandoned on the roadside? And, we are left in the thick flood of numbers and gigabit analysis data.

"One always returns to where one has started."

--Denise Levertov(1923-1997)

Hence, in dynamics, we return to the simplicity and fundamental form of d'Alembert and Lagrange.

D'Alembert-Lagrange Principal Equations

Definition:

For a N-degree of freedom system, regardless whether it is rigid or flexible, d'Alembert-Lagrange's principal equations are obtained by summing all the forces and all the moments(with respect to a point) in the system.

Symbolic Expressions:

Sum of forces (3 equations at most):

$$\Sigma(\mathbf{f}_i - m_i a_i) = 0$$

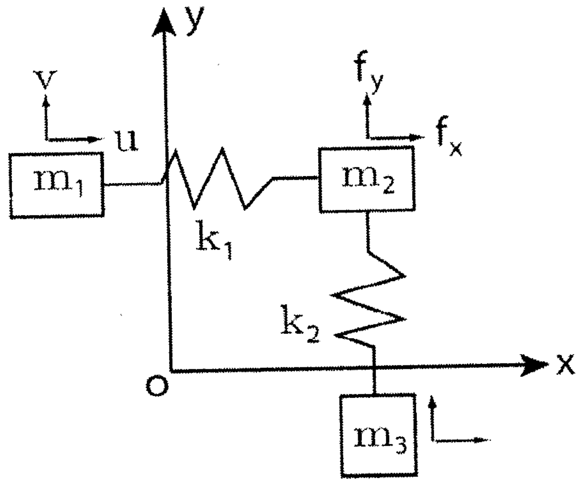
Sum of moments (3 equations at most):

$$\Sigma\{\mathbf{M}_i + \mathbf{r}_i \times (\mathbf{f}_i - m_i a_i)\} = 0$$

A bottom-up approach to

D'Alembert-Lagrange's Principal Balance Equations

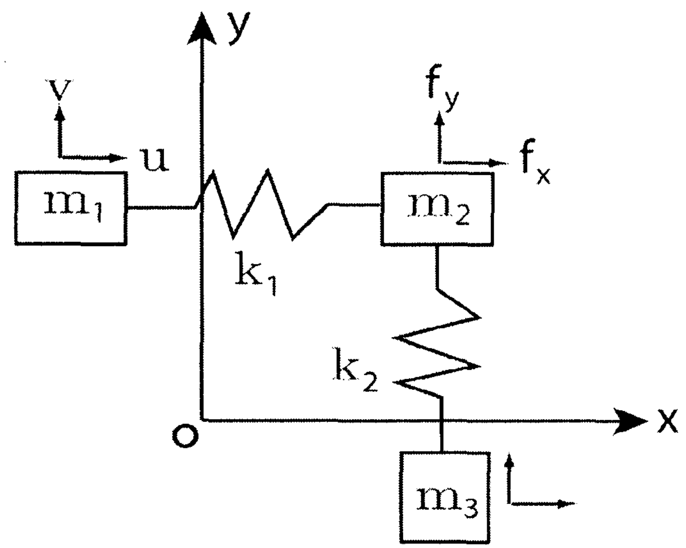
A boomerang



A 6-DOF model***

***Warning: This model does not capture the boomerang motions!

Equations of Motion for 6-DOF Boomerang Model



$$
\begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & m_2 & & & \\ & & & m_3 & & \\ & & & & m_1 & \\ & & & & & m_3 \end{bmatrix}
\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{v}_3 \\ \ddot{v}_1 \\ \ddot{u}_3 \end{Bmatrix}
$$

$$
+ \begin{bmatrix} k_1 & -k_1 & & & & \\ -k_1 & k_1 & & & & \\ & & k_2 & -k_2 & & \\ & & -k_2 & k_2 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_1 \\ u_3 \end{Bmatrix}
= \begin{Bmatrix} f_{x_1} \\ f_{x_2} \\ f_{y_2} \\ f_{y_3} \\ f_{y_1} \\ f_{x_3} \end{Bmatrix}
$$

Definition of d'Alembert's Forces

$$\mathbf{f}^D = \begin{Bmatrix} f_{x_1}^D \\ f_{x_2}^D \\ f_{y_2}^D \\ f_{y_3}^D \\ f_{y_1}^D \\ f_{x_3}^D \end{Bmatrix} = \begin{Bmatrix} f_{x_1} \\ f_{x_2} \\ f_{y_2} \\ f_{y_3} \\ f_{y_1} \\ f_{x_3} \end{Bmatrix} - \begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & m_2 & & & \\ & & & m_3 & & \\ & & & & m_1 & \\ & & & & & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{v}_3 \\ \ddot{v}_1 \\ \ddot{u}_3 \end{Bmatrix}$$

$$- \begin{bmatrix} k_1 & -k_1 & & & & \\ -k_1 & k_1 & & & & \\ & & k_2 & -k_2 & & \\ & & -k_2 & k_2 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_1 \\ u_3 \end{Bmatrix}$$

Sum of x-direction forces:

$$f_{x_1}^D + f_{x_2}^D + f_{x_3}^D = (f_{x_1} + f_{x_2} + f_{x_3}) - (m_1\ddot{u}_1 + m_2\ddot{u}_2 + m_3\ddot{u}_3) = 0$$

$$\Downarrow$$

$$\sum_{k=1}^{3} f_{x_k}^D = f_{x_1}^D + f_{x_2}^D + f_{x_3}^D = f_x - M\ddot{u}_c = 0$$

Sum of y-direction forces

$$f_{y_1}^D + f_{y_2}^D + f_{y_3}^D = (f_{y_1} + f_{y_2} + f_{y_3}) - (m_1\ddot{v}_1 + m_2\ddot{v}_2 + m_3\ddot{v}_3) = 0$$

$$\Downarrow$$

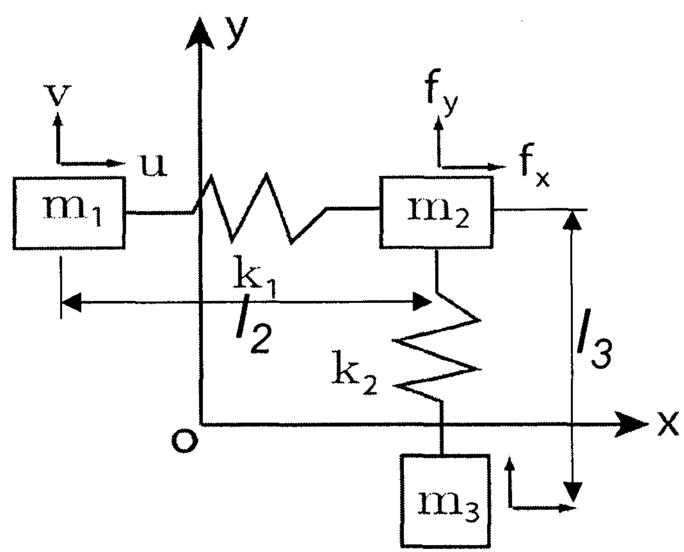
$$\sum_{k=1}^{3} f_{y_k}^D = f_{y_1}^D + f_{y_2}^D + f_{y_3}^D = f_y - M\ddot{v}_c = 0$$

Sum of moments around $(x_1 \;,\; y_1)$

$$M_{x_1}^D = \ell_2\{(f_{y_2} + f_{y_3}) - (m_2\ddot{v}_2 + m_3\ddot{v}_3)\} + \ell_3\{f_{x_3} - m_3\ddot{u}_3\} = 0$$

$$\Downarrow$$

$$M_{x_1}^D = \sum_{k=1}^{3} \begin{bmatrix} -y_k & x_k \end{bmatrix} \begin{Bmatrix} f_{x_k}^D \\ f_{y_k}^D \end{Bmatrix} = 0$$



Come on, K. C. Park! Every undergraduate knows this stuff .
And don't waste my time here in a nice place, please.

A top-down generalization via
Operational Form of Previous Equations
(We do not teach undergraduates this way!)

Sum of x-directional forces:

$$\boxed{\mathbf{S}_x^T\ \mathbf{f}^D = 0}, \qquad \mathbf{S}_x = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{f}^D = \begin{Bmatrix} f_{x_1}^D \\ f_{y_1}^D \\ f_{x_2}^D \\ f_{y_2}^D \\ f_{x_3}^D \\ f_{y_3}^D \end{Bmatrix}$$

Sum of y-directional forces:

$$\boxed{\mathbf{S}_y^T\ \mathbf{f}^D = 0}, \qquad \mathbf{S}_y^T = [0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1]$$

Sum of moment at x_1 :

$$\boxed{\mathbf{S}_{\theta_z}\ \mathbf{f}^D = 0}, \quad \mathbf{S}_{\theta_z}^T = [-y_1 \quad x_1 \quad -y_2 \quad x_2 \quad -y_3 \quad x_3]$$

D'Alembert-Lagrange Principal Equations for 6-DOF Boomerang Model

$$\mathbf{S}^T\, \mathbf{f}^d = 0$$

$$\mathbf{S} = [\,\mathbf{S}_x \quad \mathbf{S}_y \quad \mathbf{S}_{\theta_z}\,] = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ 1 & 0 & -y_3 \\ 0 & 1 & x_3 \end{bmatrix}$$

What are they?

Where does one utilize them for?

What added value do they offer, if any?

What are they?

The summation operator below

$$\mathbf{S} = [\,\mathbf{S}_x \quad \mathbf{S}_y \quad \mathbf{S}_{\theta_z}\,] = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ 1 & 0 & -y_3 \\ 0 & 1 & x_3 \end{bmatrix}$$

consists of three rigid-body modes.

Mathematically, $\mathbf{S}^T \mathbf{f}^D = \mathbf{0}$ is a projection of d'Alembert's forces onto the rigid-body modes; hence, the resulting equilibrium equations are nothing but rigid-body equations!

What are they? - Cont'd

The d'Alembert-Lagrange principal equations,

$$S^T f^D = 0,$$

represent the mean motions of flexible dynamical systems for which the instantaneous mass center of the total system is given, for the example problem, by

$$u_c = (m_1 u_1 + m_2 u_2 + m_3 u_3)/M, \quad M = (m_1 + m_2 + m_3)$$

so that the corresponding x-direction equation is given by

$$S^T{}_x f_x{}^D = f_x - M (d^2 / d t^2) u_c = 0$$

What about the principal moment equation?
(What are they? - Cont'd)

The acceleration at each mass point, $(\ddot{u}_k, \ddot{v}_k)$, can be expressed in terms of the acceleration of the instantaneous mass center, $(\ddot{u}_c, \ddot{v}_c)$, plus its mean angular acceleration, $\ddot{\theta}_c$, at the mass center:

$$\begin{Bmatrix} \ddot{u}_k \\ \ddot{v}_k \end{Bmatrix} = \begin{Bmatrix} \ddot{u}_c \\ \ddot{v}_c \end{Bmatrix} + \begin{Bmatrix} -y_k \\ x_k \end{Bmatrix} \ddot{\theta}_c$$

Since the moment due to the translational inertia forces can be shown to be

$$\mathbf{S}_{\theta_z}^T \begin{Bmatrix} m_1 \ddot{u}_1 \\ m_1 \ddot{v}_1 \\ m_2 \ddot{u}_2 \\ m_2 \ddot{v}_2 \\ m_3 \ddot{u}_3 \\ m_3 \ddot{v}_3 \end{Bmatrix} = J_c \ddot{\theta}$$

$$J_c = m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2)$$

where $m_1 x_1 + m_2 x_2 + m_3 x_3 = 0, \, etc.$

the principal moment balance equation can be shown as

$$T_c - J_c \ddot{\theta} = 0$$

$$T_c = \mathbf{S}_{\theta_z}^T \, \mathbf{f}$$

$$\mathbf{f}^T = [\, f_{x_1} \quad f_{y_1} \quad f_{x_2} \quad f_{y_2} \quad f_{x_3} \quad f_{y_3} \,]$$

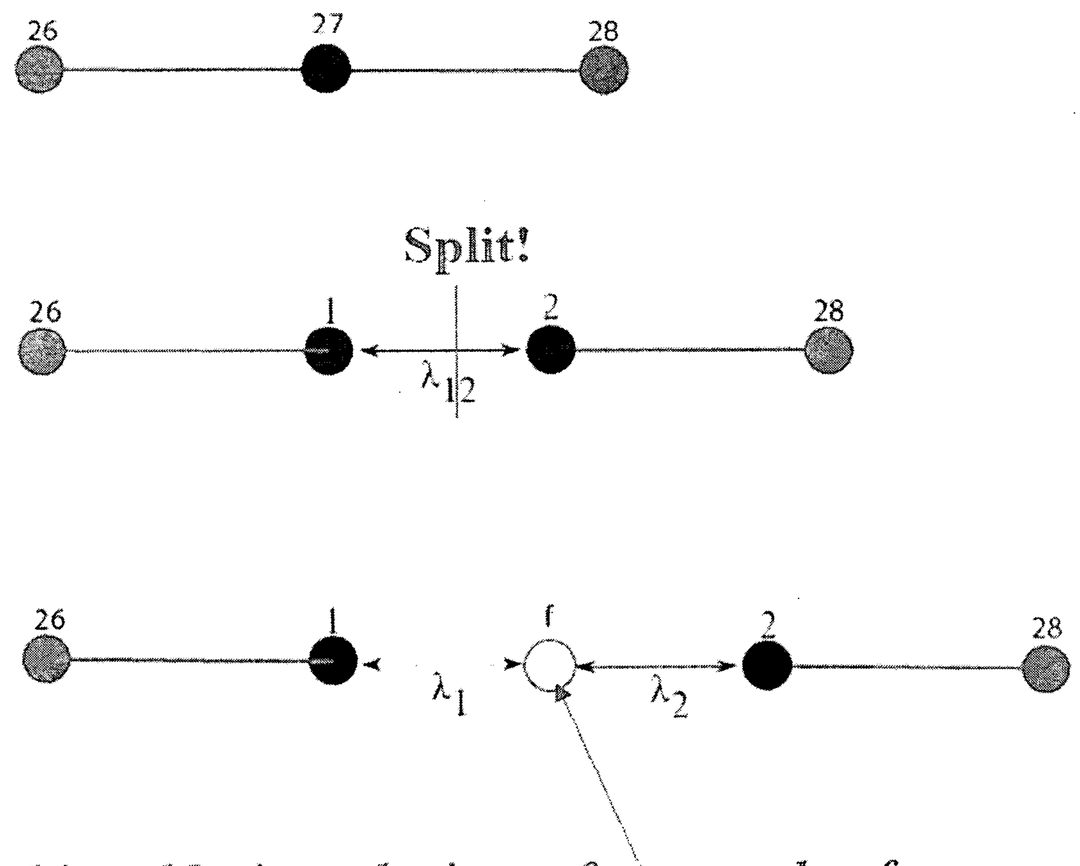
What are they for partitioned systems?

Localization of Classical Lagrange Multipliers

Assembled

Classical Partitioning

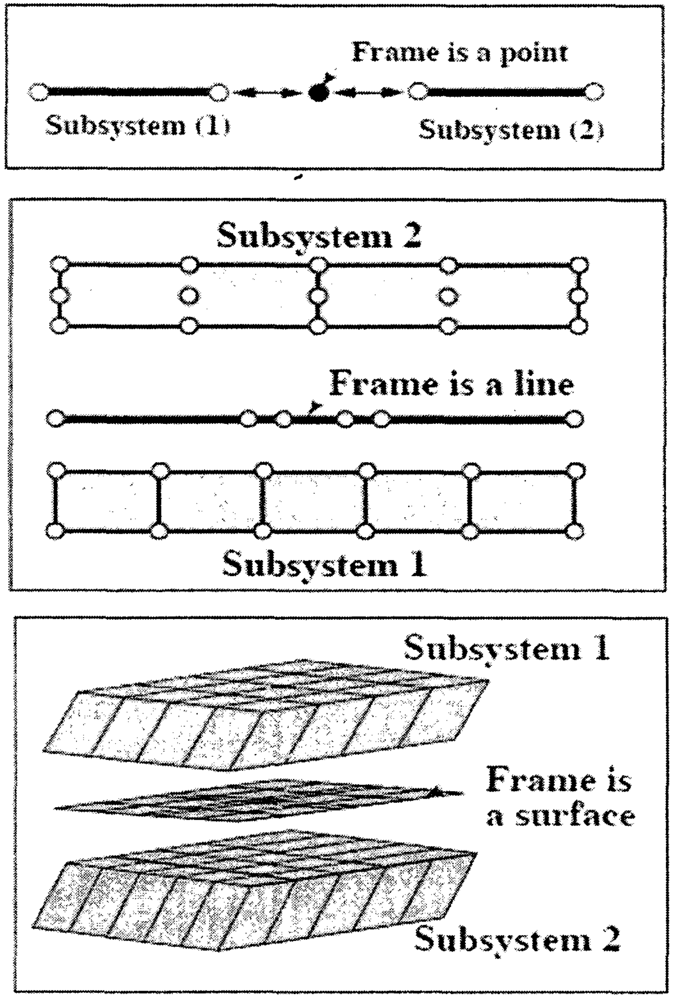
Localized Partitioning



*Localization is achieved by introducing a frame node, **f***

What are they for partitioned systems?

Localization Frame Features



What are they for partitioned systems? - Concluded.

For the partitioned modeling of flexible mechanical systems,
$S^T f^D = 0$ provides self-equilibrium condition for each partition.

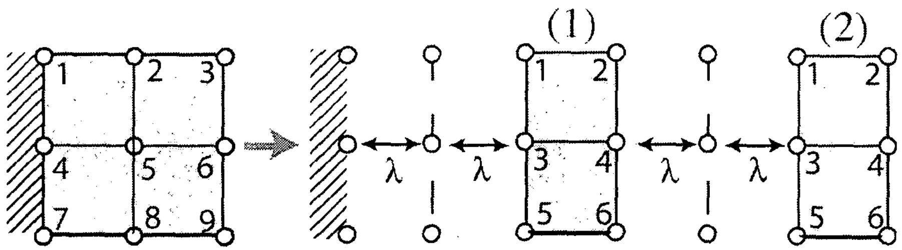
Self-equibrium equations for the two partitioned systems:

For partition 1:

$$\mathbf{S}_1^T(\mathbf{f}_1 - \mathbf{M}_1\ddot{\mathbf{u}}_1 - \mathbf{K}_1\mathbf{u}_1 - \mathbf{B}_1\boldsymbol{\lambda}_1) = 0$$

For partition 2:

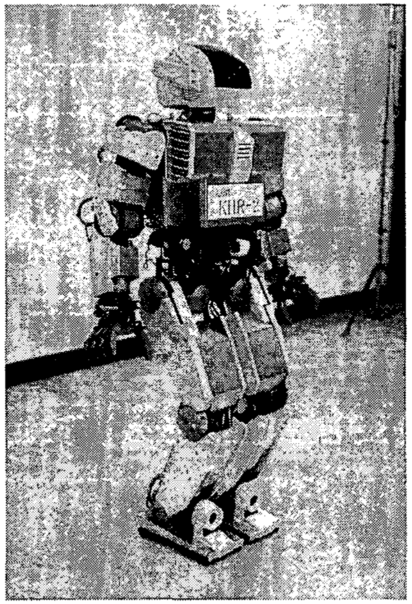
$$\mathbf{S}_2^T(\mathbf{f}_2 - \mathbf{M}_2\ddot{\mathbf{u}}_2 - \mathbf{K}_2\mathbf{u}_2 - \mathbf{B}_2\boldsymbol{\lambda}_2) = 0$$

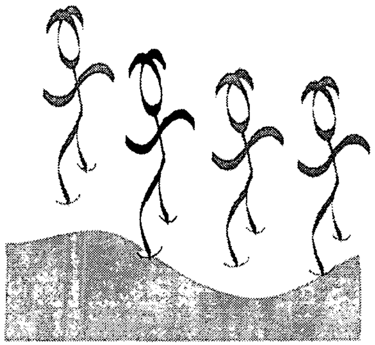


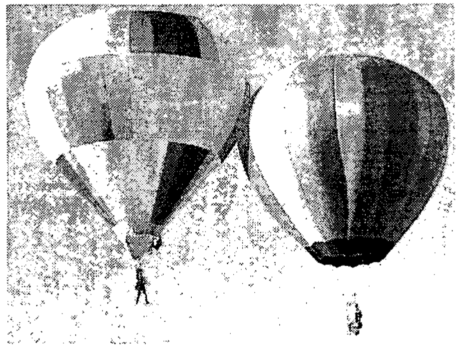
(a) Assembled structure

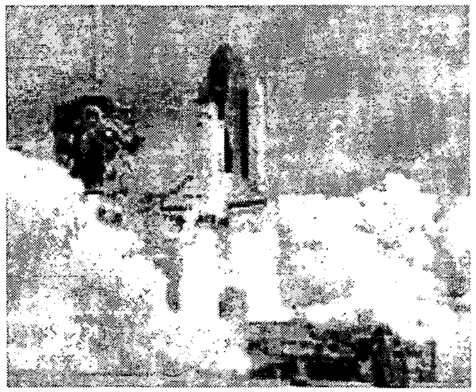
(b) Partition into two substructures

Examples for which self-equilibrium conditions are important:









Where does one utilize them for?

1. If the applied forces are known, they provide the mean motions of the system.

2. If part or all of the forcing functions are unknown and the mean motions are measured, they provide a least-squares solution of the applied forces and moments. For the example problem, they can provide a least-squares estimate of aerodynamic forces acting on the boomerang.

3. From the theoretical point of view, the d'Alembert-Lagrange principal equations provide the solvability conditions for completely free or partially constrained flexible systems, either quasi-static or dynamic. We will examine this aspect later in the lecture.

What added value do they offer, if any?

1. **In the modeling of multi-physics problems, they provide the principal (rigid-body modes) interface forces and moments, viz., average interface forces and moments.**

2. **In multi-body dynamics, if properly formulated and implemented, they provide the fundamental rigid-body motions and the corresponding joint forces that can aid first-hand physical insight for subsequent optimization, control and baseline solutions for detailed analysis.**

3. **They facilitate the divide-and-conquer paradigm for the modeling and solution of complex systems: a key property for partitioned modeling and analysis.**

4. **In the iterative solution of large-scale problems, they provide a crucial starting vector and subsequent filtering of residuals for faster iterative convergence.**

D'Alembert-Lagrange Principal Equations for General 3-D Systems (Let's get serious on their usage)

Step 1: Variational Statement of the d'Alambert-Lagrange Principal Equations:

$$\mathbf{S}^T \mathbf{f}^D = 0 \quad \longleftrightarrow \quad \delta\alpha^T \mathbf{S}^T \mathbf{f}^D = \delta \mathbf{u}_r^T \mathbf{f}^D = 0$$

$$\delta \mathbf{u}_r = \mathbf{S}\delta\alpha$$

where $\delta \mathbf{u}_r$ is clearly the virtual displacement.

When the force and moment summation operator, $\bar{S}^T$, for a floating flexible body is transposed, it becomes the rigid-body modes designated as R in the present paper.

The preceding statement is now mathematically expressed as

$$\mathbf{R} \Leftarrow \mathbf{S}$$

$$\Downarrow$$

$$\delta \mathbf{u}_r = \mathbf{R}\delta\alpha$$

Let's get serious on their usage - cont'd

The preceding variational observation allows us to decompose the displacement as

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_d, \quad \mathbf{u}_d = \Phi \mathbf{q}$$
$$\mathbf{u}_r = \mathbf{R}\alpha, \quad \mathbf{R} = \mathbf{S}$$

where $\mathbf{R}$ and Φ are the rigid body and deformational modes, respectively, and α, $\mathbf{q}$ the corresponding amplitudes.

Step 2: The summation operator for 3-dimensional problem:
For each discrete nodal point we have

$$\mathbf{S}^T = [\ \mathbf{S}_{trans} \quad \mathbf{S}_{rot}\ ]^T = [\ \mathbf{S}_1^T \quad \mathbf{S}_2^T \quad \ldots \quad \mathbf{S}_n^T\ ]$$

$$\mathbf{S}_i^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -(z_i - z_0) & (y_i - y_0) & 1 & 0 & 0 \\ (z_i - z_0) & 0 & -(x_i - x_0) & 0 & 1 & 0 \\ -(y_i - y_0) & (x_i - x_0) & 0 & 0 & 0 & 1 \end{bmatrix}$$

Variational formulation of partitioned system

$$\delta\Pi(\mathbf{u}_g) = \delta\mathbf{u}_g^T \cdot \mathbf{f}_g^D$$

$$\mathbf{f}_g^D = \mathbf{f}_g - \mathbf{K}_g\mathbf{u}_g - \mathbf{M}_g\ \ddot{\mathbf{u}}_g$$

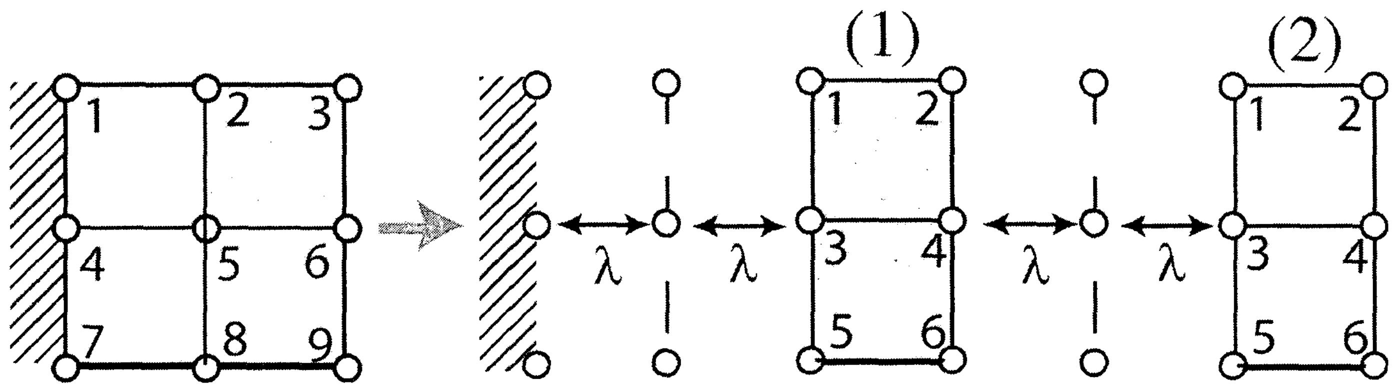
$$\mathbf{K}_g = \mathbf{L}^T\ \mathbf{K}\ \mathbf{L}, \quad \mathbf{M}_g = \mathbf{L}^T\ \mathbf{M}\ \mathbf{L}$$

where $\mathbf{u}_g$ is the discrete global displacement vector, $\mathbf{f}_g^D$ is d'Alembert's force, $\mathbf{K}_g$ and $\mathbf{M}_g$ are the assembled global stiffness and mass matrices, $\mathbf{K}$ and $\mathbf{M}$ are the block diagonal substructure-by-substructure stiffness and mass matrices, $\mathbf{L}$ is the assembly Boolean matrix, and the superscript dot $(\dot{\ })$ denotes time differentiation, respectively.

$$\begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdot \\ \cdot \\ \mathbf{u}_n \end{Bmatrix} = \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \cdot \\ \cdot \\ \mathbf{L}_n \end{bmatrix} \{\ \mathbf{u}_g\ \} \quad \Rightarrow \quad \mathbf{u} = \mathbf{L}\ \mathbf{u}_g$$

Example of Assembly Boolean Matrix

$$\mathbf{u}_l = \mathbf{L}_l \mathbf{u}_g, \qquad \mathbf{L}_l = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



(a) Assembled structure

(b) Partition into two substructures

Partition Interface constraint functional

$$\mathbf{B}^T(\mathbf{u} - \mathbf{L}\,\mathbf{u}_g) = 0 \quad \Rightarrow \quad \mathbf{B}^T\mathbf{u} - \mathbf{L}_f\,\mathbf{u}_f = 0, \quad \mathbf{L}_f = \mathbf{B}^T\mathbf{L}$$

$$\delta\pi_\ell = \lambda_\ell^T\,(\mathbf{B}^T\,\mathbf{u} - \mathbf{L}_f\,\mathbf{u}_f)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & . & . & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & . & . & \mathbf{0} \\ . & . & . & . & . \\ . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & . & . & \mathbf{B}_n \end{bmatrix}$$

where $\mathbf{u}_f$ consists of those global degrees freedom pertaining to the partition interfaces.

And B extracts the interface degrees of freedom for each partition.

Variational Statement for Partitioned Systems

$$\delta\Pi(\mathbf{u}, \mathbf{u}_f, \lambda_\ell) = \delta\mathbf{u}^T \cdot \mathbf{f}^D$$
$$+ \delta\{\lambda_\ell^T \ (\mathbf{B}^T \ \mathbf{u} - \mathbf{L}_f \ \mathbf{u}_f)\}$$
$$\mathbf{f}^D = \mathbf{f} - \mathbf{K}\mathbf{u} - \mathbf{M} \ \ddot{\mathbf{u}}$$

Introduce the decomposition of the displacement into deformation and rigid-body(D'Alembert-Lagrange principal) modes:

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_d, \quad \mathbf{u}_d = \Phi\mathbf{q}$$
$$\mathbf{u}_r = \mathbf{R}\alpha, \quad \mathbf{R} = \mathbf{S}$$

Four-Variable Variational Formulation for Partitioned Systems

$$\delta\Pi(\mathbf{u}_\alpha, \mathbf{u}_d, \mathbf{u}_f, \lambda_\ell) = \delta(\mathbf{u}_\alpha + \mathbf{u}_d)^T \cdot \mathbf{f}^D(\mathbf{u}_\alpha + \mathbf{u}_d) + \delta[\lambda_\ell^T \; (\mathbf{B}^T \; (\mathbf{u}_\alpha + \mathbf{u}_d) - \mathbf{L}_f \; \mathbf{u}_f)]$$

$$\mathbf{f}^D(\mathbf{u}_\alpha + \mathbf{u}_d) = \mathbf{f} - \mathbf{K} \; (\mathbf{u}_\alpha + \mathbf{u}_d) - \mathbf{M} \; (\ddot{\mathbf{u}}_\alpha + \ddot{\mathbf{u}}_d)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_d & \mathbf{0} & \Phi^T\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}}_\alpha & \mathbf{R}_b^T & \mathbf{0} \\ \mathbf{B}^T\Phi & \mathbf{R}_b & \mathbf{0} & -\mathbf{L}_f \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \alpha \\ \lambda_\ell \\ \mathbf{u}_f \end{Bmatrix} = \begin{Bmatrix} \Phi^T \mathbf{f} \\ \mathbf{R}^T\mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

$$\bar{\mathbf{K}}_d = \Phi^T\mathbf{M}\Phi D^2 + \Phi^T\mathbf{K}\Phi, \quad D = \frac{d}{dt}$$

$$\bar{\mathbf{K}}_\alpha = \mathbf{R}^T\mathbf{M}\mathbf{R} \; D^2$$

$$\mathbf{R}_b = \mathbf{B}^T\mathbf{R}$$

D'Alembert-Lagrange Principal Equations for Partitioned Systems

$$\mathbf{M}_\alpha \frac{d^2\alpha}{dt^2} = \mathbf{R}^T(\mathbf{f} - \mathbf{B}\lambda)$$

$$\mathbf{L}_f^T \lambda_\ell = 0$$

$$\mathbf{M}_\alpha = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} M_{tt} & M_{tr} \\ M_{tr}^T & M_{rr} \end{bmatrix}, \quad \mathbf{R} \Leftarrow \mathbf{S}$$

$$\alpha^T = \begin{bmatrix} \alpha_x & \alpha_y & \alpha_z & \alpha_{\theta_x} & \alpha_{\theta_y} & \alpha_{\theta_z} \end{bmatrix}$$

In the above equation, M_{tt} is in general a (3x3)-diagonal matrix that represents the sum of the elemental translational nodal masses for each partition, M_{rr} represents a (3x3)-rotatory inertia matrix that is usually fully coupled unless the coordinates chosen for generating $\mathbf{R}$ coincide with the principal axes of the partitioned substructures, and, M_{tr} represents the coupling between the translational mass and the rotatory mass matrix, respectively.

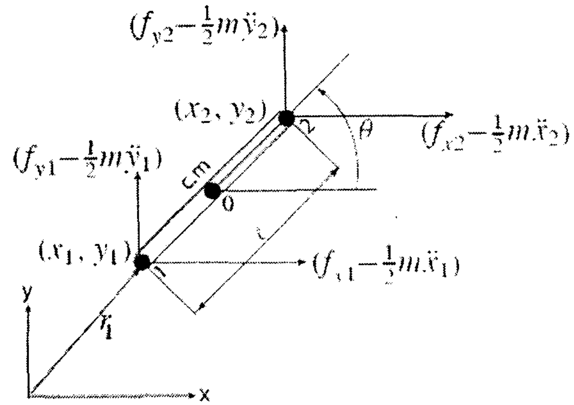
This equation provides the mean gross motions of the total system!

Application 1: mass properties of complex structures (Example - Bar element)

It is not trivial at all for computing the rotatory inertia and its coupling matrix with the translational mass properties!

Translational mass matrix of a bar element:

$$\mathbf{m}_{bar} = m \begin{bmatrix} 1/3 & 0 & 1/6 & 0 \\ 0 & 1/3 & 0 & 1/6 \\ 1/6 & 0 & 1/3 & 0 \\ 0 & 1/6 & 0 & 1/3 \end{bmatrix}$$



Summation operator for the case when the moment is computed around node 1

$$\mathbf{S}_1^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -\ell/\sqrt{2} & \ell/\sqrt{2} \end{bmatrix}$$

Application 1: mass properties of complex structures - Cont'd (Bar element example)

$$\mathrm{M}_{\alpha 1} = \mathrm{S}_1^T \mathrm{m}_{bar} \mathrm{S}_1 = m \begin{bmatrix} 1 & 0 & -\ell/2\sqrt{2} \\ 0 & 1 & \ell/2\sqrt{2} \\ -\ell/2\sqrt{2} & \ell/2\sqrt{2} & \ell^2/3 \end{bmatrix}$$

The above matrix correctly captures the rotatory inertia when the rotational motion is measured with respect to node 1 and the corresponding coupling matrix with the Translational mass. Complex cases yield the correct mass properties. Computations of the rotatory inertia for complex systems, e.g., automobile, satellites, etc., with the present procedure is straightforward and consistent.

All one needs for computing the inertia properties of complex systems are the translational mass matrix and the moment summation operator!

Application 2: Solvability for unconstrained systems under quasi-static equilibrium states

Partitioned equations of motion for structures:

$$\begin{bmatrix} \Phi^T \mathbf{K} \Phi & \mathbf{0} & \Phi^T \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^T \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T \Phi & \mathbf{B}^T \mathbf{R} & \mathbf{0} & -\mathbf{L}_f \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \alpha \\ \lambda_\ell \\ \mathbf{u}_f \end{Bmatrix} = \begin{Bmatrix} \Phi^T \mathbf{f} \\ \mathbf{R}^T \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

This equation is singular and consequently cannot be solved!

Application 2: Solvability for unconstrained systems under quasi-static equilibrium states

Eliminate q to obtain:

$$\begin{bmatrix} \mathbf{F}_{bb} & -\mathbf{R}_b & \mathbf{L}_f \\ -\mathbf{R}_b^T & 0 & 0 \\ \mathbf{L}_f^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_\ell \\ \alpha \\ \mathbf{u}_f \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}_\lambda \\ \mathbf{b}_\alpha \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}^T\mathbf{K}^+\mathbf{f} \\ -\mathbf{R}^T\mathbf{f} \\ 0 \end{Bmatrix}$$

$$\mathbf{F}_{bb} = \mathbf{B}^T\mathbf{P}_\alpha\mathbf{K}^+\mathbf{P}_\alpha\mathbf{B} = \mathbf{B}^T\mathbf{K}^+\mathbf{B}$$

$$\mathbf{u}_d = \Phi\mathbf{q} = \mathbf{P}_\alpha\mathbf{u}$$

$$\mathbf{P}_\alpha = \mathbf{I} - \mathbf{R}(\mathbf{R}^T\mathbf{R})^{-1}\mathbf{R}^T$$

This equation is regular and consequently can be solved!

Application 3: Consistent starting vector for an iterative solution of partitioned system equations

Step 1: Solve for a least-square value of the interface forces:

$$\begin{bmatrix} \mathbf{R}_b^T \\ \mathbf{L}_f \end{bmatrix} \lambda_\alpha = \begin{Bmatrix} \bar{\mathbf{b}}_\alpha \\ \mathbf{0} \end{Bmatrix}$$

$$\Downarrow$$

$$\lambda_\alpha = \mathbf{P}_L \mathbf{R}_b (\mathbf{R}_b^T \mathbf{P}_L \mathbf{R}_b)^{-1} \bar{\mathbf{b}}_\alpha$$

$$\mathbf{P}_L = \mathbf{I} - \mathbf{L}_f (\mathbf{L}_f \mathbf{L}_f^T)^{-1} \mathbf{L}_f$$

Application 3: Consistent starting vector for an iterative solution of partitioned system equations

Step 2: Project the new iterate to be orthogonal to the interface rigid-body modes:

$$\lambda_\ell = \lambda_\alpha + \mathbf{P}\triangle\lambda_\ell$$

$$\mathbf{P} = \mathbf{P}_L - \mathbf{P}_L\mathbf{R}_b(\mathbf{R}_b^T\mathbf{P}_L\mathbf{R}_b)^{-1}\mathbf{R}_b^T\mathbf{P}_L$$

$$\mathbf{P}_L = \mathbf{I} - \mathbf{L}_f(\mathbf{L}_f\mathbf{L}_f^T)^{-1}\mathbf{L}_f$$

in order to minimize the residual:

$$\mathbf{r} = \mathbf{P}(\hat{\mathbf{b}}_\lambda - \mathbf{F}_{bb}\mathbf{P}\triangle\lambda_\ell)$$

$$\hat{\mathbf{b}}_\lambda = \bar{\mathbf{b}}_\lambda - \mathbf{F}_{bb}\lambda_\alpha$$

Future Potential Applications

A. Least-squares nominal interface forces that may provide a preliminary design modification or control strategy for systems with constraints.

B. Augmented solution of (q, α for dual control strategy development, i.e., for principal motions and deformational motions in tandem.

C. Filtering of mean motion signals from output signals.

D. Advanced multi-physics modeling

Discussions

1. The d'Alembert-Lagrange principal equations consists of 6 rigid-body motions regardless how large the flexible mechanical structural systems may be, and they provide the mean motions of the overall system dynamics.

2. The rotatory inertia and its coupling terms with the translational mass properties are obtained as part of the derivational process of the d'Alembert-Lagrange principal equations presented herein.

3. The d'Alembert-Lagrange principal equations constitute the key solvability condition for systems partially constrained or in completely free-free state.

4. For an iterative solution of coupled multi-physics problems, the solution of the d'Alembert-Lagrange principal equations provides a consistent starting vector, thus accelerating the iterative process.

5. There remains a challenge to expand the usage of the d'Alembert-Lagrane principal equations, some of which have been outlined herein.

Viva la dynamique élémentaire!

Fin!

PS: More Specific Application Examples Will be Presented at the Meeting.