Bivariate Failure Modeling

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ABSTRACT: People frequently discuss equipment behavior in terms of time(age) and usage(mileage). Common examples are automobiles in which time and usageare usually included in discussion of longevity. In this paper a structured examination of bivariate measures of equipment utilization is performed and some useful model forms are developed and evaluated.

1. INTRODUCTION

In general age and mileage are not the only two quantities that might describe device longevity. We use the terms time and usage in generic terms and may represent quite different measures. In the example of an automobile tire, age might correspond to accumulated mileage and usage might be measured as tread loss. The point is that device life is a resource that may be best represented and for which the comsumption may best be measured using a two (or higher) dimensional vector and the quantities that comprise the vector are specific to the equipment.

In this paper a structured examination of bivariate reliability models is performed is developed. The paper is structured as follows. In section 2 literature review is done. In section 3 bivariate reliability models are constructed. The models examined are those in which the two variables are related by a stochastic function and those in which the variables are simply correlated. In section 4, definition and interpretation of bivariate probabilities are given so that we can better understand the models proposed in section 3. In section 5, the concepts such as cumulative distribution function, reliability function and hazard function that are defined in section 4 are illustrated to the example models. In section 6, conclusion is given along with future research opportunities.

2. LITERATURE REVIEW

Most of the previously developed bivariate reliability models treat the two variables as functionally related. Many of the models of wear process(e.g., Mercer[1961] and Lemoine and Wenocur [1985]) and several of those for cumulative damage (e.g., Barlow and Proschan [1975] and Birnbaum and Saunders [1969]) portray equipment reliability in terms of deterministically defined deterioration occurring at random points in time.

The analytical emphasis with the models based on stochastic functions has also been their reduction to a single dimension – time. The stochastic wear models and cumulative damage models that treat damage magnitude as a random variable are all defined in a manner that permits focus on reliability in time. Even the diffusion process models (Cox [1962]) that are expressed comprehensively in terms of both variables are analyzed in terms of first passage time to a failure state.

However, there are several papers that address bivariate and multivariate reliability models in a very different context than the one treated here. Specifically, Marchall and Olkin [1967a, 1967b] developed multivariate models for the reliability of series systems comprised of non-independent components. In the construction of the models.

3. BIVARIATE FAILURE MODELING

3.1 *Notation*

T	time to failure
U	usage to failure
g(t)	function relating usage and time
α,β,γ	parameters of the function g(t)
$\pi_\alpha(\cdot)$	density on the prameter α
$\lambda(t),\eta(u)$	time and usage functions that determine the failure hazard
ρ	correlation coefficient
$f_{T,U}(t,u), F_{T,U}(t,u)$	bivariate failure density and distribution functions
$\bar{F}_{T,U}(t,u)$	bivariate reliability function

3.2 *Example failure models*

3.2.1 Stochastic functions

We consider four example forms here:

(i) $g(t)=\alpha t+\beta$

(ii) $g(t)=\alpha t^2+\beta t+\gamma$

(iii) $g(t)=\alpha t^n$

We introduce randomness into the function by treating the parameter α as a random variable having distribution $\pi_\alpha(\cdot)$. This poses random variation on the extent of usage experienced by any time. The use of the distribution $\pi_\alpha(\cdot)$ to construct the marginal probability distribution on usage is accomplished using a transformation of variables. In general

Once the marginal distribution of usage is obtained we can construct the joint failure density using the

So $\quad f_U(u) = \frac{1}{t}\pi_\alpha(\frac{u-\beta}{t})$

conditioning relation

$$f_{T,U}(t\,|\,u) = f_{T|U}(t\,|\,u) f_U(u) \qquad (1)$$

and the conditional density $f_{T|U}(t\,|\,u)$ is obtained by using the well-known relationship between a density and its hazard function:

$$f_{T|U}(t\,|\,u) = z_{T|U}(t\,|\,u) e^{[-\int_0^t z_{T|U}(t|u)dt]}$$

$$= z_{T|g(T)}(t\,|\,u) e^{[-\int_0^t z_{T|g(T)}(t|u)dt]} \qquad (2)$$

We assume that the conditional hazard function on time given usage may be stated as

$$z_{T|U}(t\,|\,u) = \lambda(t) + \eta(u) \qquad (3)$$

so that the definitions of the functions $\lambda(t)$, $\eta(u)$ and $g(t)$ determine the conditional hazard and ultimately the bivariate life distribution. Here we assume simple linear functions $\lambda(t) = \lambda t$ and $\eta(u) = \eta u$ and focus on the function $g(t)$. Under this modeling format, the bivariate life distribution corresponding to form (3) above is obtained by

$$z_{T|U}(t\,|\,u) = \lambda t + \eta g(t) \qquad (4)$$

and applying (1) and (2) to obtain

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t} \exp\left\{-\frac{\eta(u+\beta)}{2}t - \frac{\lambda}{2}t^2\right\} \pi_\alpha\left(\frac{u-\beta}{t}\right) \qquad (5)$$

The same analytical approach yields the following:

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t^2} \exp\left\{-\frac{\eta(u+2\gamma)}{3}t - \frac{3\lambda+\eta\beta}{6}t^2\right\} \pi_\alpha\left(\frac{u-\beta t-\gamma}{t^2}\right) \qquad (6)$$

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t^n} \exp\left\{-\frac{\eta u}{n+1}t - \frac{\lambda}{2}t^2\right\} \pi_\alpha\left(\frac{u}{t^n}\right) \qquad (7)$$

$$f_{T,U}(t,u) = \frac{(1+\beta)(\lambda t + \eta u)}{t(1-u)(1+\beta u)} \exp\left\{-\frac{\lambda}{2}t^2 + \frac{\eta}{\beta}t - \left(\frac{\eta\frac{\beta+1}{\beta}}{\ln\frac{1+\beta u}{1-u}} \ln\frac{1}{1-u}\right)t\right\} \pi_\alpha\left(\frac{1}{t}\ln\frac{1+\beta u}{1-u}\right) \qquad (8)$$

for cases (ii), (iii), (iv) respectively.

3.2.2 Correlation

The first is the generalization of the bivariate exponential model defined by Baggs and Nagagaja [1996]. In this model, the reliability is

$$\overline{F}_{T,U}(t,u) = e^{-(\lambda t+\eta u)}(1+\rho(1-e^{-\lambda t})(1-e^{-\eta u})) \qquad (9)$$

so the corresponding density function is

$$f_{T,U}(t,u) = \lambda\eta e^{-(\lambda t+\eta u)}(1+\rho(1-2e^{-\lambda t}-2e^{-\eta u}+4e^{-(\lambda t+\eta u)})) \qquad (10)$$

the joint hazard function is

$$z_{T,U}(t,u) = \frac{\lambda\eta(1+\rho(1-2e^{-\lambda t}-2e^{-\eta u}+4e^{-(\lambda t+\eta u)}))}{(1+\rho(1-e^{-\lambda t}-e^{-\eta u}+e^{-(\lambda t+\eta u)}))} \qquad (11)$$

and the marginal densities are

$$f_T(t) = \lambda e^{-\lambda t} \, and \, f_u(u) = \eta e^{-\eta u}$$

which are both constituent exponentials regardless of the value of ρ.

A second model is the bivariate Normal. The density function for this model is well known to be

$$f_{T,U}(t,u) = \frac{1}{2\pi\sigma_t\sigma_u\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(t-\mu_t)^2}{\sigma_t^2} - 2\rho\frac{(t-\mu_t)(u-\mu_u)}{\sigma_t\sigma_u} + \frac{(u-\mu_u)^2}{\sigma_u^2}\right]\right\} \qquad (12)$$

The marginal densities are Normal.

One final model is the one stated by Hunter [1974a] in a queueing context but also consistent with reliability interpretations:

$$f_{T,U}(t,u) = \frac{\lambda\eta}{1-\rho} I_0\left(\frac{2\sqrt{\rho}}{1-\rho}\sqrt{\lambda\eta t u}\right) \exp\left\{-\frac{\lambda t+\eta u}{1-\rho}\right\} \qquad (13)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of order n; and ρ is positive. The marginal densities are

$$f_T(t) = \lambda e^{-\lambda t} \text{ and } f_U(u) = \eta e^{-\eta u}$$

4. MODEL ANALYSIS

4.1 *Bivariate probability distributions*

First we interpret the cumulative failure probability $F_{T,U}(t,u)$ as the probability that failure occurs by time t and usage u, that is;

$$F_{T,U}(t,u) = \Pr[T \le t \; and \; U \le u] \qquad (14)$$

In addition to equation (14) the rectangles are $\Pr[T \le t, U > u]$, $\Pr[T \le t, U < u]$, and $\Pr[T \ge t, U > u]$.

It is not obvious whether the following cumulative probabilities

$$\Pr[T \le t, U > u] = \int_0^t \int_u^\infty f_{T,U}(s,v)dvds \qquad (15)$$

and

$$\Pr[T > t, U \le u] = \int_t^\infty \int_0^u f_{T,U}(s,v)dvds \qquad (16)$$

are survival probabilities. So we call them marginal survival probabilities.

A further point is the fact that the reliability at (t,u)

does not include the marginal survival probabilities (15) and (16). The reliability at (t,u) is as follows:

$$\bar{F}_{T,U}(t,u)=\Pr[T\ge t,U\ge u]=\int_t^\infty\int_u^\infty f_{T,U}(s,v)dvds \quad (17)$$

The apparent paradox is that $F_{T,U}(t,u)\neq 1-\bar{F}_{T,U}(t,u)$. For any rectangle, say $[t_1\le T\le t_2, u_1\le U\le u_2]$ in the plane, the probability of observing a failure at a point in the rectangle is:

$$\Pr[t_1\le T\le t_2,u_1\le U\le u_2]=F_{T,U}(t_2,u_2)$$
$$-F_{T,U}(t_2,u_1)-F_{T,U}(t_1,u_2)+F_{T,U}(t_1,u_1) \quad (18)$$

4.2 *Hazard fuctions*

$$z_{T,U}(t,u)$$
$$=\lim_{\substack{\Delta t\to 0\\ \Delta u\to 0}}\frac{\Pr\left[t\le T\le t+\Delta t,u\le U\le u+\Delta u\,|\,T>t,U>u\right]}{\Delta u\Delta t}$$
$$=\lim_{\substack{\Delta t\to 0\\ \Delta u\to 0}}\frac{\Pr\left[t\le T\le t+\Delta t,u\le U\le u+\Delta u\right]}{\Delta u\Delta t\Pr\left[T>t,U>u\right]}$$
$$=\frac{1}{\bar{F}_{T,U}(t,u)}\lim_{\substack{\Delta t\to 0\\ \Delta u\to 0}}\frac{F_{T,U}(t+\Delta t,u+\Delta u)-F_{T,U}(t+\Delta t,u)-F_{T,U}(t,u+\Delta u)+F_{T,U}(t,u)}{\Delta u\Delta t}$$
$$=\frac{1}{\bar{F}_{T,U}(t,u)}\frac{\partial^2}{\partial t\partial u}F_{T,U}(t,u)$$
$$=\frac{f_{T,U}(t,u)}{\bar{F}_{T,U}(t,u)} \quad (19)$$

By analogy with the univariate case, one may want to construct bivariate models given the bivariate hazard. For the univarite case, this approach may be accomplished using the relationship:

$$z(t)=\ln\bar{F}(t) \quad \text{or} \quad \bar{F}(t)=e^{-z(t)}$$

4.3 *Moments*

The moment generating function for the bivariate failure distribution is:

$$M_{T,U}(\theta_1,\theta_2)=E\left[e^{\theta_1 t+\theta_2 u}\right]=\int_0^\infty\int_0^\infty e^{\theta_1 t+\theta_2 u}f_{T,U}(t,u)dudt \quad (20)$$

and the moments of the distribution are obtained as:

$$E\left[t^k\right]=\frac{\partial^k}{\partial\theta_1^k}M_{T,U}(\theta_1,\theta_2)\Big|_{\substack{\theta_1=0\\ \theta_2=0}} \quad \text{and}$$

$$E\left[u^k\right]=\frac{\partial^k}{\partial\theta_2^k}M_{T,U}(\theta_1,\theta_2)\Big|_{\substack{\theta_1=0\\ \theta_2=0}} \quad (21)$$

4.4 *Renewal functions*

$$f_{T,U}^{(k)}(t,u)=\int_0^t\int_0^u f_{T,U}^{(k-1)}(t-s,u-v)f_{T,U}(s,v)dvds \quad (22)$$

The definition and interpretation of the associated counting process and the bivariate renewal function is less obvious and may depend upon the application.

5. EXAMPLE CALCULATIONS OF BIVARIATE FAILURE MODELS

In the case of stochastic function models assume $\pi_\alpha(\cdot)$ is a negative exponential density of the form

$$\pi_\alpha(\cdot)=ce^{-c\alpha} \quad (23)$$

Then with $\lambda=10^{-6}$, β=10, $\eta=1.5*10^{-6}$, c=1000, equation (8) yields the following values for the cumulative distribution function as in Table 1.

Table 1. CDF values for equation (8)

Corresponding reliability values are shown in Table 2.

The hazard values are shown in Table 3.

6. CONCLUSIONS

In this paper the models examined were those in which the two variables are related by a stochastic function and those in which the variables are simply correlated. The concepts such as cumulative distribution function, reliability function and hazard function were illustrated to the example models.

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