Boundary Control of a Tensioned Elastic Axially Moving String

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Abstract: In this paper, an active vibration control of a tensioned elastic axially moving string is investigated. The dynamics of the translating string are described by a non-linear partial differential equation coupled with an ordinary differential equation. A time varying control in the form of right boundary transverse motions is proposed in stabilizing the transverse vibrations of the translating continuum. A control law based on Lyapunov's second method is derived. Exponential stability of the closed-loop system is verified. The effectiveness of the proposed controller is shown through simulations.

Keywords: Axially moving string, boundary control, exponential stability, hyperbolic partial differential equation, Lyapunov method, nonlinear string

1. INTRODUCTION

The control problems of axially moving systems occur in various engineering areas: For example, the strips in thin metal-sheet production lines, the cables, belts, and chains in power transmission lines, the magnetic tapes in recorders, the band saws, etc. The dynamics of these systems can be differently modeled depending on the length, flexibility, and control objectives of the system considered. For instance, the dynamics of a moving cable of an elevator can be described by a string equation, but that of a rubber belt in the traditional mill can be well represented by a belt equation. The difference between a string and a belt lies in whether the longitudinal elongation is considered or not.

In axially moving systems, the transverse (lateral) vibration of the moving material often causes a serious problem in achieving good quality. It is also known that these vibrations are often caused by the eccentricity of a pulley, and/or an irregular speed of the driving motor, and/or a non-uniform material property, and/or environmental disturbances. Since the quality requirement as well as the productivity in a production line is getting stricter, an active or a semi-active vibration control is nowadays seriously considered.

In this paper, the vibration control method to reduce the vibration which occurs during a continuous hot-dip zinc galvanizing process is considered. In order to achieve the uniformity of the zinc deposit on the strip surfaces and to reduce the zinc consumption, the strip should pass at equidistance from each of the air knives. But, due to the shifting and vibration of the strip, a discrepancy between the averaged deposited masses on the left and right strip surfaces and a non-uniformity of the deposited mass across the strip occur. These variations in deposited mass will degrade the quality of the product.

Depending on the thickness of the strip and the distance between two support points, Axially moving system can be modeled as one out of three models: a moving beam, a moving string, and a moving belt. In the zinc galvanizing line, the distance between two-support points is quite large compared to the strip thickness and width is small. Therefore, the modeling as a string is invested. In the given system, in-flux and out-flux mass exist and at the time varying boundary which the control force is given by the moving mass work is occurred. So we use Hamilton's principle for system of changing mass to derive nonlinear equation of motion.

The previous researches about axially moving string didn't consider the elasticity of the string and used more than two sensors. Lee and Mote [4] derived an optimal boundary force control law that dissipates the vibration energy of an axially moving string and, in [5], analyzed the wave characteristics of the beam and derived optimal boundary damping laws as a function of linear velocity, linear slope, and linear force.

The contributions of this paper are the following. First, boundary control law about non-linear string to consider the constant tension and elasticity of the string is derived. Second, the derived boundary control law requires only one sensor to apply itself. Additionally, the damping coefficient of the actuator is designed.

2. EQUATIONS OF MOTION

Fig. 1 shows a schematic (model) of the axially moving string considered, which will be used in deriving equations of motion and a boundary control law. The string is assumed to travel at a constant speed. The left boundary is fixed in the sense that the boundary itself does not have any vertical (transversal) movement, but it allows the material to move longitudinally. However, the right boundary permits a transversal movement of the string under a control force and is time varying.

Let t be the time, x be the spatial coordinate along the longitude of motion, v be the axial speed of the string, $w(x,t)$ be the transversal displacement of the string at spatial coordinate x and time t, and L be the length of the string. Then, the absolute velocity of the string at spatial coordinate x is given by

$$\overline{v} = v\,\mathrm{i} + Dw(x,t)/Dt\ \mathrm{j} = v\,\mathrm{i} + \{w_t(x,t) + vw_x(x,t)\}\mathrm{j}\,, \qquad (1)$$

where $D(\cdot)/Dt \overset{\Delta}{=} \partial(\cdot)/\partial t + v\partial(\cdot)/\partial x$, and $(\cdot)_t = \partial(\cdot)/\partial t$, $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives in time t and spatial coordinate x, respectively.

The kinetic energy of the axially moving string including the actuator, and the potential energy are written as follows:

$$T = \frac{1}{2}\int_0^L \rho A\left\{v^2 + \left(w_t + vw_x\right)^2\right\}dx + \frac{1}{2}mw_t^2(L,t)\,, \qquad (2)$$

$$U = \int_0^L \left(P_o\varepsilon_x + \frac{EA}{2}\varepsilon_x^{\ 2}\right)dx\,, \qquad (3)$$

where T is the kinetic energy, U is the potential (strain) energy, ρ is the mass per unit volume (material density), A is the cross-sectional area, m is the mass of the actuator

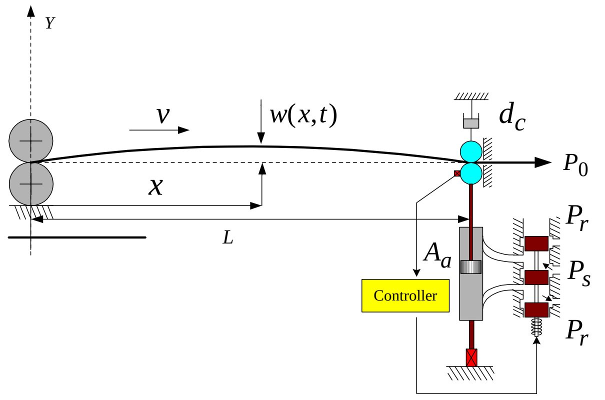

Fig. 1 An axially moving strip under a right boundary control force.

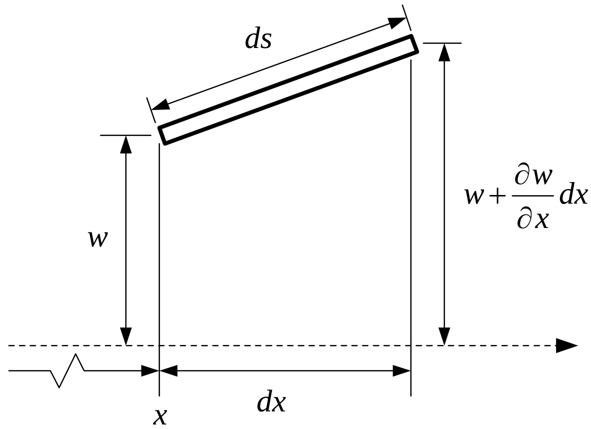

Fig. 2 Schematic of string elongation for small slopes.

(touch roll), E is the elastic modulus of the string, ε_x is the strain due to the tension P_0. The potential energy is proportional to the increase in string length ds when compared to the string at rest. For small slopes, see Fig. 2, the following relationship for string elongation is valid:

$$\varepsilon_x = \frac{ds - dx}{dx} = \left[1 + \left(\frac{\partial w(x,t)}{\partial x}\right)^2\right]^{1/2} - 1 \cong \frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 . \quad (4)$$

Therefore, (3) is rewritten as follows,

$$U = \frac{1}{2}\int_0^L \left(P_0 + \frac{EA}{4} w_x^2\right) w_x^2 dx \ . \quad (5)$$

Now, to derive the equations of motion, Hamilton's principle for the systems with changing mass is utilized as follows:

$$\delta \int_{t_1}^{t_2} \left(T - U + W_{n.c.} + W_{r.b.}\right) dt = 0 , \quad (6)$$

where $W_{n.c.}$ is the non-conservative work, $W_{r.b.}$ is the virtual momentum transport at the right boundary (no variations at the left boundary). The variations of the non-conservative work and the virtual momentum transport at the right boundary are

$$\delta W_{n.c.} = F_c(t)\delta w(L,t) - d_c w_t(L,t)\delta w(L,t) , \quad (7)$$

$$\delta W_{r.b.} = -\rho A v\{w_t(L,t) + v w_x(L,t)\}\delta w(L,t) , \quad (8)$$

where $F_c(t)$ is the control force, and d_c is the damping coefficient of the actuator. Now, the variations of (2) and (5), respectively, are

$$\delta T = \rho A \int_0^L (w_t + v w_x)(\delta w_t + v \delta w_x) dx$$
$$+ m w_t(L,t)\delta w_t(L,t) , \quad (9)$$

$$\delta U = P_0 \int_0^L w_x \delta w_x dx + \frac{EA}{2}\int_0^L w_x^3 \delta w_x dx \ . \quad (10)$$

The substitution of (7)-(10) into (6) yields

$$\int_{t_1}^{t_2} \left(\delta T - \delta U + \delta W_{n.c} + \delta W_{r.b}\right) dt$$
$$= \int_{t_1}^{t_2}\int_0^L \rho A (w_t + v w_x)\delta w_t \, dx\, dt$$
$$+ \int_{t_1}^{t_2}\int_0^L \left(\rho A v w_t + \rho A v^2 w_x\right)\delta w_x \, dx\, dt$$
$$+ \int_{t_1}^{t_2}\int_0^L \left(-P_0 w_x - \frac{EA}{2} w_x^3\right)\delta w_x \, dx\, dt$$
$$+ \int_{t_1}^{t_2} \left[m w_t(L,t)\delta w_t(L,t)\delta w(L,t)\right] dt$$
$$+ \int_{t_1}^{t_2} \{F_c(t) - d_c w_t(L,t)\}\delta w(L,t)\, dt$$
$$- \int_{t_1}^{t_2} \left\{\rho A v w_t(L,t) + \rho A v^2 w_x(L,t)\right\}\delta w(L,t) = 0 . \quad (11)$$

And the integration of (11) by parts yields

$$\int_0^L \left[(\rho A w_t + \rho A v w_x)\delta w\right]_{t_1}^{t_2} dx$$
$$- \int_{t_1}^{t_2}\int_0^L (\rho A w_{tt} + \rho A v w_{xt})\delta w \, dx\, dt$$
$$+ \int_{t_1}^{t_2} \left[\left(\rho A v w_t + \rho A v^2 w_x\right)\delta w\right]_0^L dt$$
$$- \int_{t_1}^{t_2}\int_0^L \left(\rho A v w_{tx} + \rho A v^2 w_{xx}\right)\delta w dx dt$$
$$+ \int_{t_1}^{t_2} \left[\left(-P_0 w_x - \frac{EA}{2} w_x^3\right)\delta w\right]_0^L dt$$
$$- \int_{t_1}^{t_2}\int_0^L \left(-P_0 w_{xx} - \frac{3EA}{2} w_x^2 w_{xx}\right)\delta w dx dt$$
$$+ \left[m w_t(L,t)\delta w(L,t)\right]_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \{m w_{tt}(L,t) - F_c(t) + d_c w_t(L,t)\}\delta w(L,t) dt$$
$$- \int_{t_1}^{t_2} \left\{\rho A v w_t(L,t) + \rho A v^2 w_x(L,t)\right\}\delta w(L,t) dt . \quad (12)$$

Note that $\delta w(0,t) = 0$ because the left end is fixed (i.e., $w(0,t) = 0$). Therefore (12) is rewritten as follows:

$$- \int_{t_1}^{t_2}\int_0^L (\rho A w_{tt} + 2\rho A v w_{tx})\delta w dx dt$$
$$+ \int_{t_1}^{t_2}\int_0^L \left(P_0 - \rho A v^2 + \frac{3EA}{2} w_x^2\right) w_{xx}\delta w dx dt$$
$$- \int_{t_1}^{t_2} \{m w_{tt}(L,t) - F_c(t) + d_c w_t(L,t)\}\delta w(L,t) dt$$
$$- \int_{t_1}^{t_2} \left\{P_0 + \frac{EA}{2} w_x^2(L,t)\right\} w_x(L,t)\, \delta w(L,t) dt = 0 . \quad (13)$$

Since δw is arbitrary except for the requirement that the left end is fixed (i.e., $\delta w(0,t) = 0$), the following governing equation and a boundary constraint at the right end are derived as follows:

$$\rho A w_{tt} + 2\rho A v w_{xt} - \left(P_0 - \rho A v^2 + \frac{3EA}{2} w_x^2\right) w_{xx} = 0 , \quad (14)$$

$$F_c(t) = m w_{tt}(L,t) + d_c w_t(L,t)$$

$$+\left\{P_0+\frac{EA}{2}w_x^2(L,t)\right\}w_x(L,t)\,, \tag{15}$$

where w_{tt} is the local acceleration in the transversal direction of the string, w_{xt} is the Coriolis' acceleration, and $v^2 w_{xx}$ is the centripetal acceleration.

Remark: For a linear system, which is the case that $3EAw_x{}^2/2=0$, the solution of (14) can be obtained through the method of separation of variables. In this case, the natural frequency is given by $\omega_n=\frac{n\pi}{cL}(c^2-v^2),\, n=1,2,3,\cdots$, where $c=\sqrt{P_0/\rho A}$ is called the wave velocity (see [8]). The natural frequency decreases as the traveling speed increases. If the traveling speed is equal to the wave velocity, the natural frequency becomes zero and a divergence of the solution occurs. In this sense, c is called the critical speed v_{cr}. Hence, the following is also assumed in this paper.

$$0<v<v_{cr}=\sqrt{P_0/\rho A}\,. \tag{16}$$

If using the parameters in Table 1, we can calculate $v_{cr}=\sqrt{P_0/\rho A}=$ 407.99 m/sec.

3. BOUNDARY CONTROL LAW

In this section, we design a boundary controller including a damper to reduce the transversal vibration of a longitudinally moving elastic string.

A positive definite function in the form of total mechanical energy of the string excluding the actuator is first considered as follows:

$$V_s(t)=\frac{1}{2}\int_0^L \rho A(w_t+vw_x)^2 dx$$
$$+\frac{1}{2}\int_0^L\left(P_0+\frac{EA}{8}w_x^2\right)w_x^2 dx\,, \tag{17}$$

where the subscript s stands for string.

Lemma 1: Consider a functional $\tilde{V}$

$$\tilde{V}(t)=V_s(t)+V_c(t)\,, \tag{18}$$

with the second (complementary) term is defined by

$$V_c(t)=\beta\rho A\int_0^L xw_x(w_t+vw_x)dx\,, \tag{19}$$

where $\beta>0$ is a constant. Then (17) and (18) are equivalent, that is, there exist constants $\beta>0$ and $0<C_1<1$ satisfying

$$(1-C_1)V_S(t)\le\tilde{V}(t)\le(1+C_1)V_S(t)\,. \tag{20}$$

Proof: The existence of such β and C_1 will be proved. First, (19) becomes

$$V_c(t)=\rho A\beta\int_0^L xw_x(w_t+vw_x)dx$$
$$\le\frac{\rho A\beta L}{2}\left\{\int_0^L w_x^2dx+\int_0^L(w_t+vw_x)^2dx\right\}$$
$$\le\rho A\beta L\left\{\frac{1}{P_0}\cdot\frac{P_0}{2}\int_0^L w_x^2dx+\frac{1}{2}\int_0^L(w_t+vw_x)^2dx\right\}$$
$$+\rho A\beta L\left\{\frac{1}{EA}\cdot\frac{EA}{8}\int_0^L w_x^4dx\right\}$$
$$\le C_1V_S(t)\,, \tag{21}$$

where

$$C_1=\frac{\rho A\beta L}{\min(P_0,\rho A,EA)}>0\,. \tag{22}$$

Hence the following holds:

$$-C_1V_s(t)\le V_c(t)\le C_1V_s(t)\,. \tag{23}$$

By adding $V_s(t)$ at both sides of (23), we obtain

$$(1-C_1)V_S(t)\le\tilde{V}(t)\le(1+C_1)V_S(t)\,. \tag{24}$$

In order to $1-C_1>0$, the range of β is restricted by

$$0<\beta<\frac{\min(P_0,\rho A,EA)}{\rho AL}\,. \tag{25}$$

Verifying the range of β using the parameters in Table 1, $0<\beta<1/L=0.05$ is obtained. In this paper, $\beta=$ 0.03 is selected and $C_1=0.6$ is calculated. Lemma 1 is proved. ■

Now, with Lemma 1, the following Lyapunov function candidate $V(t)$, which is basically equivalent to the total mechanical energy of the string and the actuator, is proposed.

$$V(t)=\tilde{V}(t)+V_a(t)\,, \tag{26}$$

where the additional actuated-related term $V_a(t)$ is defined as

$$V_a(t)=\frac{m}{2}\{w_t(L,t)+(v+\beta L)w_x(L,t)\}^2\,. \tag{27}$$

To derive the time derivative of (26), a fixed control volume is introduced, as in Fig. 3. Volume II represents the part of the string that occupies the inner part of the control volume at an arbitrary time t, whereas I and III represent the influx and efflux of the string at $t+dt$, respectively. Using Reynolds transport theorem, the time-derivative of (26) is given by

$$dV(t)/dt=\partial V/\partial t+v\partial V/\partial x\,\Big|_0^L\,. \tag{28}$$

The first term in the right-hand side of (28) means the time rate of the equivalent energy within the control volume and the second term is the net energy flux into the control volume.

Noting that V includes V_s in (17), V_c in (19) and V_a in (27), individual terms are evaluated as follows:

$$\partial V_s(t)/\partial t=\int_0^L\rho A(w_t+vw_x)(w_{tt}+vw_{xt})dx$$
$$+\int_0^L\left(P_0+\frac{EA}{2}w_x^2\right)w_xw_{xt}dx$$
$$=(P_0-\rho Av^2)[w_tw_x]_0^L+\frac{v(P_0-\rho Av^2)}{2}[w_x^2]_0^L$$
$$+\frac{EA}{2}[w_x^3w_t]_0^L-\frac{\rho Av}{2}[w_t^2]_0^L+\frac{3EAv}{8}[w_x^4]_0^L\,, \tag{29}$$

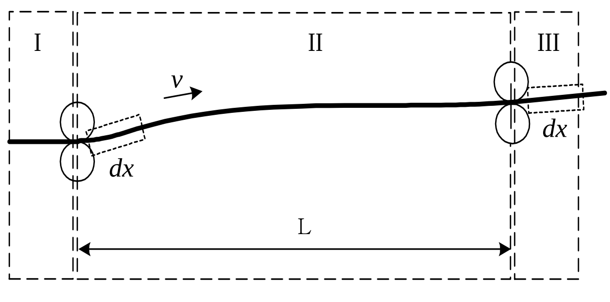

Fig. 3 Control volume of an axially moving string system with a time varying right boundary.

$$v\,\partial V_s(t)/\partial x\Big|_0^L = v\int_0^L \rho A(w_t + vw_x)(w_{tx} + vw_{xx})dx$$
$$+ v\int_0^L P_0 w_x w_{xx} dx + \frac{EAv}{2}\int_0^L w_x^3 w_{xx} dx$$
$$= \frac{\rho A v}{2}\Big[(w_t + vw_x)^2\Big]_0^L + \frac{vP_0}{2}\Big[w_x^2\Big]_0^L + \frac{EAv}{8}\Big[w_x^4\Big]_0^L . \quad (30)$$

Using (29) and (30), $dV_s(t)/dt$ becomes

$$dV_s(t)/dt = \left(P_0 - \rho A v^2\right)\left[w_t w_x\right]_0^L + \frac{v\left(P_0 - \rho A v^2\right)}{2}\Big[w_x^2\Big]_0^L$$
$$+ \frac{EA}{2}\Big[w_x^3 w_t\Big]_0^L - \frac{\rho A v}{2}\Big[w_t^2\Big]_0^L + \frac{3EAv}{8}\Big[w_x^4\Big]_0^L$$
$$+ \frac{\rho A v}{2}\Big[(w_t + vw_x)^2\Big]_0^L + \frac{vP_0}{2}\Big[w_x^2\Big]_0^L + \frac{EAv}{8}\Big[w_x^4\Big]_0^L$$
$$= P_0 w_x(L,t) w_t(L,t) + vP_0\left\{w_x^2(L,t) - w_x^2(0,t)\right\}$$
$$+ \frac{EA}{2} w_x^3(L,t) w_t(L,t) + \frac{EAv}{2}\left\{w_x^4(L,t) - w_x^4(0,t)\right\}. \quad (31)$$

For $V_c(t)$, the following are derived:

$$\partial V_c(t)/\partial t = \rho A\beta\int_0^L x w_{xt}(w_t + vw_x)\,dx$$
$$+ \rho A\beta\int_0^L x w_x (w_{tt} + vw_{xt})dx , \quad (32)$$
$$v\,\partial V_c(t)/\partial x = v\rho A\beta\int_0^L w_x(w_t + vw_x)dx$$
$$+ v\rho A\beta\int_0^L x w_{xx}(w_t + vw_x)dx$$
$$+ v\rho A\beta\int_0^L x w_x(w_{tx} + vw_{xx})dx . \quad (33)$$

Using (32) and (33), $dV_c(t)/dt$ becomes

$$dV_c(t)/dt = v\rho A\beta\int_0^L (x w_{xt} w_x + x w_t w_{xx} + w_x w_t)dx$$
$$+ \rho A v^2\beta\int_0^L x w_x w_{xx} dx + \rho A\beta\int_0^L x w_{xt} w_t dx$$
$$+ \beta\int_0^L x w_x\left(\rho A w_{tt} + 2\rho A v w_{xt} + \rho A v^2 w_{xx}\right)dx$$
$$+ \rho A v^2\beta\int_0^L w_x^2 dx . \quad (34)$$

To rewrite (34), the following integrations by parts are utilized.

$$\int_0^L (x w_{xt} w_x + x w_t w_{xx} + w_x w_t)dx$$
$$= \left[x w_x w_t\right]_0^L = L w_x(L,t) w_t(L,t) , \quad (35)$$
$$\int_0^L x w_x w_{xx} dx = \frac{L}{2} w_x^2(L,t) - \frac{1}{2}\int_0^L w_x^2 dx , \quad (36)$$
$$\int_0^L x w_{xt} w_t dx = \frac{L}{2} w_t^2(L,t) - \frac{1}{2}\int_0^L w_t^2 dx . \quad (37)$$

Also, using the equation of motion (14), the following equation is derived.

$$\beta\int_0^L x w_x\left(\rho A w_{tt} + 2\rho A v w_{xt} + \rho A v^2 w_{xx}\right)dx$$
$$= \beta\int_0^L x w_x\left\{\left(P_0 + \frac{3EA}{2} w_x^2\right)w_{xx}\right\}dx . \quad (38)$$

Therefore, the substitution of (35)-(38) into (34) yields

$$dV_c(t)/dt = \beta\rho A L v w_x(L,t) w_t(L,t) + \frac{\rho A v^2 \beta L}{2} w_x^2(L,t)$$
$$- \frac{\rho A v^2\beta}{2}\int_0^L w_x^2 dx + \frac{\beta L P_0}{2} w_x^2(L,t) - \frac{\beta P_0}{2}\int_0^L w_x^2 dx$$
$$+ \frac{3\beta EAL}{8} w_x^4(L,t) - \frac{3\beta EA}{8}\int_0^L w_x^4 dx + \frac{\rho A\beta L}{2} w_t^2(L,t)$$
$$- \frac{\rho A\beta}{2}\int_0^L w_t^2 dx + \rho A v^2\beta\int_0^L w_x^2 dx . \quad (39)$$

Also, the time-derivative of (27) is

$$dV_a(t)/dt = m\left\{w_t(L,t) + (v + \beta L)w_x(L,t)\right\}$$
$$\times\left\{w_{tt}(L,t) + (v + \beta L)w_{xt}(L,t)\right\}. \quad (40)$$

Finally, the main part in this paper is stated as follows:

Theorem: Consider the following axially moving system

$$\rho A w_{tt} + 2\rho A v w_{xt} - \left(P_0 - \rho A v^2 + \frac{3EA}{2} w_x^2\right)w_{xx} = 0 ,$$
$$w(0,t) = 0 , \quad m w_{tt}(L,t) + d_c w_t(L,t)$$
$$+ \left\{P_0 + \frac{EA}{2} w_x^2(L,t)\right\}w_x(L,t) = F_c(t) ,$$
$$w(x,0) = w_0(x),\; w_t(x,0) = \dot{w}_0(x) . \quad (41)$$

If the control force $F_c(t)$ and the damping coefficient d_c are given by

$$F_c(t) = -K w_{xt}(L,t) , \quad (42)$$
$$\beta\rho A L/2 < d_c < v\rho A\beta L/(v + \beta L) , \quad (43)$$

where $K = m(v + \beta L)$ is the control gain, then the closed-loop system is exponentially stable.

Proof: The substitution of (42) into (15) yields

$$m w_{tt}(L,t) = -d_c w_t(L,t)$$
$$- \left\{P_0 + \frac{EA}{2} w_x^2(L,t)\right\}w_x(L,t) - m(v + \beta L)w_{xt}(L,t) . (44)$$

The substitution of (44) into (40) yields

$$dV_a(t)/dt = \left\{w_t(L,t) + (v + \beta L)w_x(L,t)\right\}$$
$$\times\left\{-d_c w_t(L,t) - P_0 w_x(L,t) - \frac{EA}{2} w_x^3(L,t)\right\} . \quad (45)$$

Therefore, combining (31), (39), and (45), the time-derivative of the Lyapunov function candidate (26) is given by

$$dV(t)/dt = -\frac{\beta L}{2}\left(P_0 - \rho A v^2\right)w_x^2(L,t) - \frac{\beta EAL}{8} w_x^4(L,t)$$
$$- \frac{\beta}{2}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx - \frac{\beta\rho A}{2}\int_0^L w_t^2 dx$$
$$- \frac{3\beta EA}{8}\int_0^L w_x^4 dx - vP_0 w_x^2(0,t) - \frac{EAv}{2} w_x^4(0,t)$$
$$- \left(d_c - \frac{\beta\rho A L}{2}\right)w_t^2(L,t)$$
$$+ \left\{\beta\rho A L v - d_c(v + \beta L)\right\}w_x(L,t) w_t(L,t)$$
$$\le -\frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t) - \frac{\beta EAL}{8} w_x^4(L,t)$$
$$- \frac{\beta}{2}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx - \frac{\beta\rho A}{2}\int_0^L w_t^2 dx$$
$$- \frac{3\beta EA}{8}\int_0^L w_x^4 dx - vP_0 w_x^2(0,t) - \frac{EAv}{2} w_x^4(0,t)$$
$$- \frac{1}{2}\left(d_c - \frac{\beta\rho A L}{2}\right)w_t^2(L,t) - \frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$
$$- \frac{1}{2}\left(d_c - \frac{\beta\rho A L}{2}\right)w_t^2(L,t)$$
$$+ \left\{\beta\rho A L v - d_c(v + \beta L)\right\}w_x(L,t) w_t(L,t) . \quad (46)$$

Since v is under the critical speed (see (16)), $P_0 > \rho A v^2$ is satisfied. Note that the first and eighth terms after the equality sign have been split into two halves. Note also that the first eight terms followed by the inequality sign are all negative, whereas the last three terms are combined to make them negative as a whole. The following notations are introduced.

$$\phi_1 = \beta L(P_0 - \rho A v^2)/4 > 0, \quad (47)$$

$$\phi_2 = \left(d_c - \beta \rho AL/2\right)/2, \quad (48)$$

$$\psi = \left\{\beta \rho ALv - d_c\left(v + \beta L\right)\right\}. \quad (49)$$

From (48) and (49), if d_c satisfies the range

$$d_c > \beta \rho AL/2 \Rightarrow d_c^- \text{ and}$$

$$d_c < \beta \rho ALv/\left(v + \beta L\right) \Rightarrow d_c^+, \quad (50)$$

then the last three terms in (46) satisfies the following inequality.

$$-\phi_1 w_x^2 + \psi w_x w_t - \phi_2 w_t^2 \le -\min\left\{\phi_1, \psi/2, \phi_2\right\}(w_t - w_x)^2. \quad (51)$$

Therefore, from (46) and (51), the asymptotic stability of the closed-loop system is assured.

Now, the exponential stability is shown with further manipulation of the terms. (46) can be rewritten, by splitting the third term into two parts, as

$$\begin{aligned} dV(t)/dt \le & -\frac{\beta EAL}{8} w_x^4(L,t) - vP_0 w_x^2(0,t) \\ & -\frac{EAv}{2} w_x^4(0,t) - \frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx \\ & -\frac{\beta}{4v^2}\left(P_0 - \rho A v^2\right)\int_0^L (v w_x^2) dx - \frac{\beta\rho A}{2}\int_0^L w_t^2 dx \\ & -\frac{3\beta EA}{8}\int_0^L w_x^4 dx - \frac{\beta L}{4}\left(P_0 - \rho A v^2\right) w_x^2(L,t) \\ & -\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right) w_t^2(L,t) \\ & -\min\{\phi, \psi/2, \varphi\}\left\{w_x(L,t) - w_t(L,t)\right\}^2 \\ \le & -\frac{\beta EAL}{8} w_x^4(L,t) - vP_0 w_x^2(0,t) - \frac{EAv}{2} w_x^4(0,t) \\ & -\frac{\beta\rho A}{2}\int_0^L w_t^2 dx - \frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx \\ & -\min\left[\frac{\beta\rho A}{2}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4v^2}\right]\left[\int_0^L w_t^2 dx + \int_0^L (v w_x^2) dx\right] \\ & -\frac{3\beta EA}{8}\int_0^L w_x^4 dx - \frac{\beta L}{4}\left(P_0 - \rho A v^2\right) w_x^2(L,t) \\ & -\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right) w_t^2(L,t) \\ & -\min\{\phi, \psi/2, \varphi\}\left\{w_x(L,t) - w_t(L,t)\right\}^2. \end{aligned} \quad (52)$$

By using the following inequality

$$-\int_0^L w_t^2 dx - \int_0^L (v w_x^2)\, dx \le -\frac{1}{2}\int_0^L (w_t + v w_x)^2 dx, \quad (53)$$

and eliminating the first four and last (negative) terms, (52) can be rewritten as

$$\begin{aligned} dV(t)/dt \le & -\min\left[\frac{\beta\rho A}{2}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4v^2}\right]\left[\int_0^L \left(w_t + v w_x\right)^2 dx\right] \\ & -\frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx - \frac{3\beta EA}{8}\int_0^L w_x^4 dx \\ & -\frac{\beta L}{4}\left(P_0 - \rho A v^2\right) w_x^2(L,t) - \frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right) w_t^2(L,t) \\ \le & -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2}\right] \\ & \times\left[\frac{P_0}{2}\int_0^L w_x^2 dx + \frac{EA}{8}\int_0^L w_x^4 dx + \frac{\rho A}{2}\int_0^L \left(w_t + v w_x\right)^2 dx\right] \\ & -\min\left[\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta\rho AL}{2}\right)\right] \\ & \times\left[\frac{m}{2}\left\{w_t(L,t) + \left(v + \beta L\right) w_x(L,t)\right\}^2\right]. \end{aligned} \quad (54)$$

If using (17) and (26)-(27), (54) can be expressed as

$$\begin{aligned} dV(t)/dt \le & -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2}\right]\tilde{V}(t) \\ & -\min\left[\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta\rho AL}{2}\right)\right] V_A(t) \\ \le & -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2},\right. \\ & \left.\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta\rho AL}{2}\right)\right] \\ & \times\left(\tilde{V}(t) + V_A(t)\right) = -\lambda V(t). \end{aligned} \quad (55)$$

(55) means the following relationship

$$V_{total}(t) \le V_0 e^{-\lambda t}, \quad (56)$$

where $V_0 = V(0)$, and λ is given by

$$\begin{aligned} \lambda = \min & \left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2},\right. \\ & \left.\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta\rho AL}{2}\right)\right] > 0. \end{aligned} \quad (57)$$

Therefore, all the variables included in (26) converge exponentially to zero. ■

4. IMPLEMENTATION AND SIMULATION

Implementing (42) and (43) requires two things: measurement of $w_{xt}(L,t)$ and satisfaction of the range $d_c^- < d_c < d_c^+$. Since the damping range is related to a design problem of the actuator, it must be answered ahead. Using the parameters in Table 1 with β = 0.03, (50) is verified as follows:

$$17.67 < d_c < 27.17. \quad (58)$$

Therefore, the existence of such a damping range is assured. The implementation of $w_{xt}(L,t)$ can be achieved by backwards differencing of $w_x(L,t)$ measured at each step.

To demonstrate the performance of the closed loop system, computer simulations using the finite difference scheme have been performed. The plant parameters used for simulations are gathered in Table 1.

With $\beta = 0.03$ given in (25), the control gain is calculated as follows:

$$K = m(v + \beta L) = 15(2 + 0.03 \times 20) = 39. \quad (59)$$

For simulation purpose, let $d_c = 25$. Let the initial conditions be

$$w(x,0) = 2\sin(3\pi) \text{ [cm]}, \quad w_t(x,0) = 0 \text{ [m/s]}. \quad (60)$$

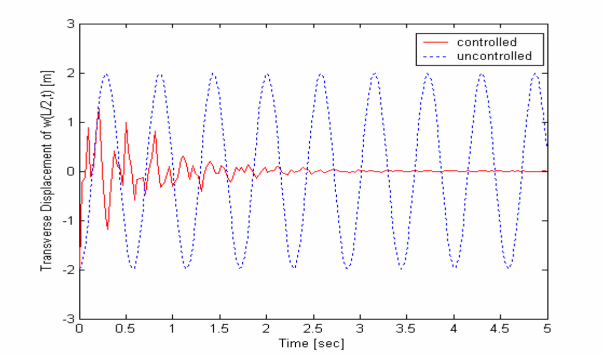

Fig. 4 The transverse displacement with control gain $K = 39$, and damping coefficient $d_c = 25$, $w(L/2, t)$ where $L = 20$ m.

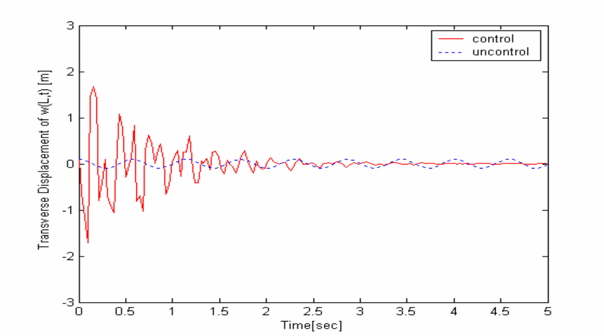

Fig. 5 The transverse displacement with control gain $K = 39$, and damping coefficient $d_c = 25$, $w(L, t)$ where $L = 20$ m.

Fig. 4 and Fig. 5 show the transverse displacement at $x = L/2$ and the control force at $x = L$, respectively. The designed controller eliminates the transversal vibration in 3 seconds. Fig. 6 shows the variation of boundary control input at $x = L$.

5. CONCLUSIONS

This paper investigated a transverse vibration suppression scheme of an axially moving non-linear string, and discussed the development of an efficient active controller and its implementation. In previous researches, the string was mostly modeled as a linear string. However, we derived a non-linear string PDE equation with actuator dynamics. The boundary control law was derived by Lyapunov method. Exponential stability of the closed-loop system was proved. The efficiency of the designed controller was shown by simulations.

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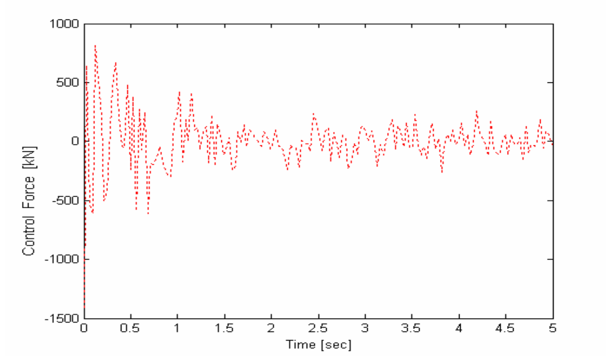

Fig. 6 The control force used in Fig. 4, 5.

Table 1 The plant parameters for simulation.

Symbols	Definitions	Values
A	Cross-section area	1.5×0.005 m^2
L	Length of string	20 m
P_0	Tension of the string	9,800 kN
m	Mass of the actuator	15 kg
v	String moving speed	2 m/s
ρ	Mass per unit area	7,850 kg/m^2
d_c	Damping coefficient	25 Ns/m