Swinging-up the Rotational Inverted Pendulum with Limited Sector of Arm Angle via Energy Control

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Abstract: Inverted pendulum is a classical example and a famous tool for testing the effectiveness of many control schemes. Owing to their nonlinearity and unstable characteristic, a controller development either for swinging-up or stabilizing its upright position had been a great interest of many researchers. In this paper, the swinging-up control of the inverted pendulum using energy control will be presented. However, the saturation function in its control law could harm the experimental equipments. In addition, this swinging-up method did not consider limited sector of the arm angle to avoid another hazard, for instance, the twisted cable in the apparatus. Therefore, in this paper the position control of the arm angle using simple PD control in accordance with the energy control is proposed. Consequently, the limited arm sector angle can be achieved and the saturation function can also be replaced effectively by the PD control.

Keywords: Rotational Inverted Pendulum, Swinging-up Control, LQR and Energy Control

1. INTRODUCTION

Swinging-up of inverted pendulum has been considered by many researchers for many years. Some swinging-up ideas recently proposed including sliding-mode control incorporated with arm position control [1], feedback linearization [2] and energy control [3]. The energy control will be adopted as an approach in this paper due to its simplicity.

The swinging-up strategy using energy control is simple. The energy is pumped to the system to increase the energy level of the pendulum. The way of pumping the energy is by giving some acceleration to the pendulum pivot i.e. the arm angle, whose direction is determined by the information of the rate of the injected energy. As the exact control law is difficult to be implemented therefore when prescribed energy level is achieved, it must be switched to the stabilizing control law to catch and stabilize the inverted pendulum in the upright position. Consequently, there will be two separated controllers e.g. the swinging-up control and the stabilizing control. In this paper, instead of using saturation function [3], the swinging-up controller using the position control of the arm angle with simple PD control in accordance with the energy control is proposed. This implies to two advantages: firstly the saturation function can be replaced properly, and secondly the limited sector of arm angle can be achieved. On the other hand, for the stabilizing controller, the LQR method [4] will be employed to find the state feedback gain matrix and the integral gain.

2. ROTATIONAL INVERTED PENDULUM

2.1 Model of the system

The rotational inverted pendulum is a SIMO system with motor torque as the input. Employing the Euler-Lagrange equation, a nonlinear model of the inverted pendulum system can be obtained. It is should be noted here that the dynamics of rotational inverted pendulum will be different to those of inverted pendulum on cart due to the effect of the rotating arm. The derivation of the rotational inverted pendulum is omitted since it can be found in many publications. In this paper, the derivation is taken from [5] by neglecting some part of the rotational inverted pendulum such as the motor dynamics, the moment of inertia of the rotating arm and the pendulum rod for the sake of simplicity. The model of the inverted pendulum shown in Fig. 1 can be described as the nonlinear differential equation in the following equations

$$\frac{d\theta}{dt} = \dot{\theta} \tag{1}$$

$$\frac{d\dot{\theta}}{dt} = \frac{1}{J + m(R^2 + l^2)\sin^2\theta}\Big[-2ml^2\sin\theta\cos\theta\dot{\beta}\dot{\theta} - mlR\sin\theta\cos^2\theta\dot{\beta}^2 - mRg\sin\theta\cos\theta + mRl\sin\theta\dot{\theta} + u\Big] \tag{2}$$

$$\frac{d\beta}{dt} = \dot{\beta} \tag{3}$$

$$\frac{d\dot{\beta}}{dt} = \frac{1}{J + m(R^2 + l^2)\sin^2\theta}\Big[\frac{J + mR^2 + ml^2\sin^2\theta}{R\cos\theta}\cdot\big(l\sin\theta\cos\theta\dot{\beta}^2 + g\sin\theta\big) - 2ml^2\sin\theta\cos\theta\dot{\beta}\dot{\theta} - mRl\sin\theta\dot{\theta}^2 - u\Big] \tag{4}$$

where θ is the pendulum angle, β is the rotating arm angle, J, m, l, R, u and g are respectively DC motor moment of inertia, inverted pendulum mass, length of the pendulum, length of the pendulum arm, input torque acting of the arm pivot and gravity acceleration.

In the compact form those can be expressed as

$$\dot{x} = f(x,u) \tag{5}$$

where $x = \begin{bmatrix}\theta & \dot{\theta} & \beta & \dot{\beta}\end{bmatrix}^T$.

2.2 Linearization

In order to design the stabilizing control described later, this nonlinear model of the rotational inverted pendulum needs to be linearized around the upright position i.e. $x = 0$ and $u = 0$ as

$$\frac{dx}{dt} = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0} + \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} = Ax + Bu \tag{6}$$

where A and B are respectively expressed as

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g(J+mR^2)}{Jl} & 0 & 0 & \frac{Rb}{Jl} \\ 0 & 0 & 0 & 1 \\ -\frac{mRg}{J} & 0 & 0 & -\frac{b}{J} \end{bmatrix} \text{ and } B=\begin{bmatrix} 0 \\ -\frac{R}{Jl} \\ 0 \\ \frac{1}{J} \end{bmatrix}.$$

It is seen that A is rank deficient and having at least one positive eignvalue. The main interest is to control the pendulum angle and the rotating arm angle. Therefore, the output equation is chosen as

$$y = Cx \tag{7}$$

where $C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T$.

Fig. 1 Experimental apparatus in laboratory.

3. CONTROLLER DESIGN

3.1 Swinging-up controller

Let's consider only the single inverted pendulum which is obtained by fixing its pivot to the arm of the inverted pendulum shown in Fig. 2. The equation of motion then can be simplified as follows

$$J_p \frac{d^2\theta}{dt^2} = \frac{1}{2}(mgl\sin\theta - mla\cos\theta) \tag{8}$$

where J_p is the moment of inertia of the pendulum and a is approximated to be the acceleration of the rotating arm or in this case

$$R\frac{d^2\beta}{dt^2} \approx a . \tag{9}$$

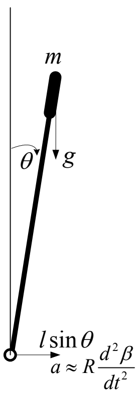

Fig. 2 Inverted pendulum when the base is fixed to its arm.

The total energy of eq. (8) is given by

$$E = \frac{1}{2}J_p\dot{\theta}^2 + mgl\cos\theta . \tag{10}$$

It should be noticed that the potential energy is chosen such that the arm position has zero energy level. In order to know that the energy is pumped correctly, we must obtain the direction whether the energy is injected or removed from the system. It can be determined from the rate of energy as

$$\dot{E} = J_p\dot{\theta}\ddot{\theta} - mgl\dot{\theta}\sin\theta = -mal\dot{\theta}\cos\theta . \tag{11}$$

Therefore, we can control the energy of inverted pendulum instead of controlling the system directly. The last equation tells us that the energy can be pumped to the system when the arm acceleration a is in the opposite direction of $\dot{\theta}\cos\theta$. If the acceleration is enough, we can drive the energy to desired level then we set acceleration to zero, at which the inverted pendulum can be swung up in one swing. If the acceleration is not enough, we can apply the maximum arm angle acceleration, however multiple swings will be needed to bring the pendulum to the inverted position. As described in [3], the appropriate control law is $u \approx mRa$, with a is chosen as

$$a = sat(\Psi(E-E_d)\cdot \operatorname{sgn}(\dot{\theta}\cos\theta)) \tag{12}$$

where Ψ is a constant gain, E_d is the desired energy level, sgn is the sign function and sat is the saturation function. Suppose that the input torque saturates to $\pm\tau_m$ which results to the saturated arm angle acceleration $\pm a_m$, with the choice of Lyapunov function candidate $V=\frac{1}{2}(E-E_d)^2$ then we can find

$$\begin{aligned}\dot{V} &= -ml(E-E_d)\dot{\theta}\cos\theta sat(\Psi(E-E_d)\dot{\theta}\cos\theta) \\ &\le -ml\left((E-E_d)\dot{\theta}\cos\theta\right)^2 \le 0\end{aligned} \tag{13}$$

which is negative semi definite. Moreover since the pendulum cannot stay identically in $\theta = \pm\frac{\pi}{2}$, then the energy must be driven toward the desired energy.

In reality this situation cannot be assured. As the energy is computed using the information of state equations, the accuracy of its information relies on the accuracy of the state variables as well and thus it is very difficult to obtain the accurate control law. Therefore, it motivates us to switch to another controller to catch the inverted pendulum when it is around the upright position.

In addition, in the control law, the saturation function is not preferred as the DC motor using maximum torque will always be applied. We also do not know how many times the pendulum arm will rotate. Thus in this paper, the swinging-up control using modified control law is proposed. Let's consider the arm angle equation of motion of the rotational inverted pendulum equation of motion in eq. (4). In the swinging-up stages we can assume that the DC motor moment of inertia is the most significant compared to the other inertia factors, therefore it can be simplified to be

$$\frac{d\dot{\beta}}{dt} = \ddot{\beta} = \frac{1}{J}(-u + b\dot{\beta}) . \tag{14}$$

Then a simple position control using proportional and derivative (PD) control can be designed by setting the control law u as follows

$$u = k_p(r-\beta) - k_d\dot{\beta} \tag{15}$$

where k_p and k_d are the proportional and the derivative gain respectively, r is the reference whose value is determined from the information of energy given by

$$r = -r_c \,\mathrm{sgn}(\dot{\theta}\cos\theta) \tag{16}$$

with r_c is positive constant. Consequently the control law proposed by [3] as shown in eq. (12) can be replaced by eqs. (15) together with (16). The structure of swinging-up control scheme is depicted in Fig. 3.

The advantage of our proposed method is the sector of the arm angle is restricted within $\pm r_c$ radian sector of arm angle. Moreover, the saturation function can be replaced effectively by the arm angle position control. Despite the swinging-up period is longer than [3] but we have a freedom in choosing the PD gains to speed up the swinging-up response.

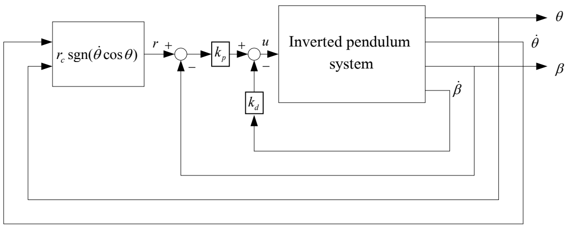

Fig. 3 Swinging-up control system structure.

3.2 Stabilizing controller

The structure of stabilizing control is depicted in Fig. 4. An integrator is augmented to the arm angle in order to reject the offset occurred in the response due to uncertainty in the hardware [4]. Let the error signal is denoted by $e(t) = r(t) - \bar{y}(t)$, then defining the new state variable $\dot{x}_i(t) = e(t)$ the augmented system can be expressed as

$$\frac{d}{dt}\begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix} = \begin{bmatrix}A & 0\\ -H & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\dot{u} \tag{17}$$

$$\dot{\bar{y}} = \begin{bmatrix}H & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix} \tag{18}$$

where $\bar{y}$ is the controlled output of the arm angle and $H = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$ which is obtained from the 2nd row of C matrix. In order to stabilize the upright position the control law

$$\dot{u} = -\begin{bmatrix}K & -k_i\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix} \tag{19}$$

that minimizes the performance index

$$J = \int_0^\infty \left\{ \begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix}^T \begin{bmatrix}Q & 0\\ 0 & q_i\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{x}_i\end{bmatrix} + \dot{u}^T R \dot{u} \right\} dt \tag{20}$$

should be found.

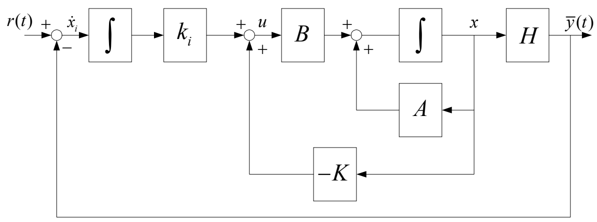

Fig. 4 Stabilizing control system structure.

Finally the optimal control law is obtained by integrating the control law (18) as follows

$$u = -Kx + k_i x_i \tag{21}$$

4. EXPERIMENTAL RESULTS

The experimental apparatus is depicted in Fig. 5. It consists of three main parts: the inverted pendulum system, the interfaces and the digital controller. The pendulum system composes of pendulum, rotating-arm, a high torque permanent magnet DC motor and two angular positions sensors to detect the pendulum angle and the arm position angle. The interface devices are two microcontrollers PIC16C55 to filter the quadrature signal from each encoder, one microcontroller 89C1051 as a sampling clock generator, one eight-bit D/A converter and servo amplifier. A personal computer with Intel Pentium II 350 MHz processor is used as the digital controller. The control program is written in C language and the sampling period is set at 25 milliseconds. The parameters of the rotational inverted pendulum are depicted in table 1.

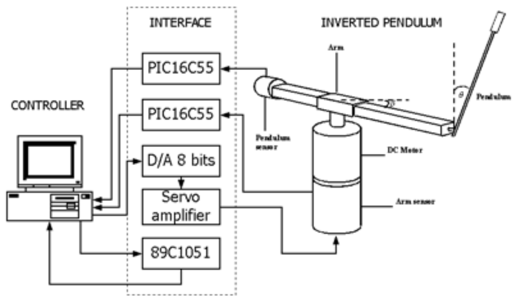

Fig. 5 Experimental apparatus.

Table 1 Parameter of the rotational inverted pendulum.

Pendulum mass (m)	0.05 kg
Pendulum length (l)	0.48 m
Arm length (R)	0.47 m
Moment of inertia (J)	0.03264 $kg \cdot m^2$
Viscous coefficient (b)	0.00351 $kg \cdot m^2 / s$

In order to swing the pendulum up as fast as possible, the response of the position control of the arm angle must also be fast enough. However, the choice PD gains to make unnecessarily fast response of the position control will make the position control being superior over the energy control. It means that the energy cannot be pumped to the system even though the direction of energy has not changed its sign.

Based on those considerations, in the experiment the proportional gain and the derivative gain using trial and error are found to be -1.3056 and -0.3299 respectively. The reference constant is set to be 1 radian and the condition for the control switching is chosen in the region of attraction of the stabilizing controller around 0.1 radian. For the stabilizing controller using Q=diag[0.8 0 0.4 0] and R=1, the optimal state feedback gains and integral gain are respectively found as -5.6365,-1.1789, -0.8844, -0.678 and -0.3164. From Fig. 6 the swinging-up results show that the pendulum can be brought to the upright position in approximately 4 seconds and the arm angle can also be brought back to zero radian line. Moreover, during the swinging-up period the arm angle is bounded within 2 radians sector of arm angle. The control signal is shown in Fig. 7. It is also seen that during swinging-up stage the control signal is peaked in several intervals implies to energy saving.

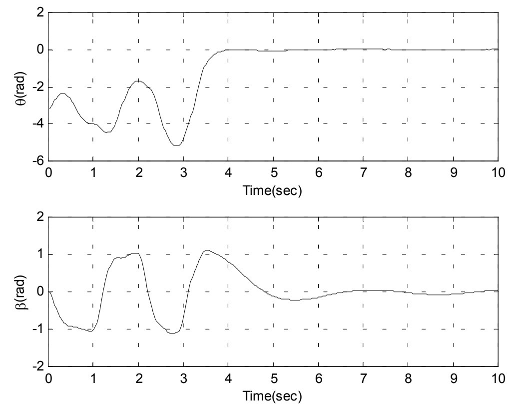

Fig. 6 Responses of the system.

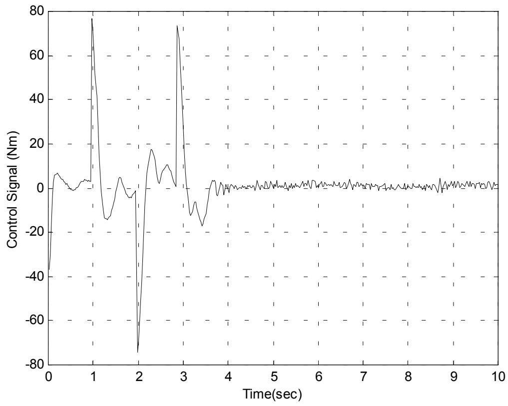

Fig. 7 Control signal.

5. CONCLUSION

Swinging-up control using energy based approach with restricted sector of arm angle is studied in this paper. The position control using proportional plus derivative scheme is proposed to replace the saturation function. The experimental results show that the pendulum can be swung-up to the inverted position effectively and its arm angle is also bounded in certain sector of angle.

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