Robust non-fragile H_∞ control of singular systems

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Abstract: This paper considers the synthesis of non-fragile H_∞ state feedback controllers for singular systems and static state feedback controller with multiplicative uncertainty. The sufficient condition of controller existence, the design method of non-fragile H_∞ controller, and the measure of non-fragility in controller are presented via LMI(linear matrix inequality) technique. Also, through singular value decomposition, some changes of variables, and Schur complements, the sufficient condition can be rewritten as LMI form in terms of transformed variables. Therefore, the obtained non-fragile H_∞ controller guarantees the asymptotic stability and disturbance attenuation of the closed loop singular systems within a prescribed degree. Finally, a numerical example is given to illustrate the design method.

Keywords: singular systems, non-fragile control, H_∞ control, LMI, uncertainty

1. INTRODUCTION

It is generally known that feedback system designed for robustness with respect to plant parameters, or designed for the optimization of a single performance measure, may require very accurate controllers. An implicit assumption that is inherent to those control methodologies is that the controller is designed to be implemented precisely. However, the controller implementation is subject to A/D conversion, D/A conversion, finite word length and round-off errors in numerical computations, in addition to the requirement of providing the practicing engineer with safe-tuning margins. Therefore, it is necessary that any controller should be able to tolerate some uncertainty in parameters. Since controller fragility is basically the performance deterioration of a feedback control system due to inaccuracies in controller implementation, non-fragile control problem has been important issues [1-6].

In a recent paper, Keel *et al.* [1] have shown that the resulting controllers exhibit a poor stability margin if not implemented exactly. So, some researchers have developed non-fragile controller design algorithms [2-6]. Haddad *et al.* [2] proposed a non-fragile controller design method via quadratic Lyapunov bounds. And, Famularo *et al.* [3] considered LQ robust non-fragile static state feedback controller design method in the presence of uncertainties in plant and controller. However, Famularo *et al.* [3] did not obtain the value of the non-fragility directly, but predetermine the value of non-fragility before finding a controller. Also Dorato *et al.* [4] dealt with the problem on the design of non-fragile compensators via symbolic quantifier elimination. However, the existing works have been focused on the non-singular systems.

Recently, much attention has been given to the extensions of the results of H_∞ control theory for state-space systems to singular systems. State space models are very useful, but the state variables thus introduced do not provide a physical meaning. Hence, the singular form is a natural representation of linear dynamical systems, and makes it possible to analyze a larger class of systems than state space equations do [7,8], because state space equations cannot represent algebraic restrictions between state variables and some physical phenomena, like impulse and hysterisis which are important in circuit theory, cannot be treated properly. Although H_∞ control theory in singular systems has been developed over the last decade, there are no papers considering non-fragile H_∞ controller design methods for singular systems. This is the motivation of the proposed design algorithm.

In this paper, we propose a non-fragile H_∞ state feedback controller design method for singular systems and static state feedback controller with multiplicative uncertainty. The sufficient condition of controller existence, the design method of non-fragile H_∞ controller, and the measure of non-fragility in controller are presented via LMI (linear matrix inequality) technique. The measure of non-fragility is related to the performance of controller gain variations. Also, through singular value decomposition, some changes of variables, and Schur complements, the sufficient condition can be rewritten as an LMI form in terms of transformed variables. Since the proposed controller design algorithm is an LMI form in terms of all variables, the solutions can be obtained simultaneously.

The following notations will be used in this paper.$(\cdot)^T$, $(\cdot)^{-1}$, $deg(\cdot)$, $det(\cdot)$, and $rank(\cdot)$ denote the transpose, inverse, degree, determinant, and rank of a matrix. An identity matrix with proper dimensions is denoted as I. I_r, $x_r(t)$, and $\mathbf{R}^r$ denote an identity matrix with $r\times r$ dimension, $r\times 1$ dimensional vector, and $r\times 1$ dimensional real vector, respectively. * represents the transposed elements in the symmetric positions.

2. NON-FRAGILE CONTROLLER DESIGN

Consider a singular system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t) \\ z(t) &= Cx(t) \end{aligned} \tag{1}$$

where, $x(t)\in\mathbf{R}^n$ is the state variable, $z(t)\in\mathbf{R}^l$ is the controlled output variable, $u(t)\in\mathbf{R}^m$ is the control input variable, $w(t)\in\mathbf{R}^p$ is the disturbance input variable, E is singular matrix with $rank(E)=r\le n$, and all matrices have proper dimensions. Although one finds the controller

$$u(t) = Kx(t)\ , \tag{2}$$

The actual controller implemented is assumed as

$$u(t) = [I + \alpha\Phi(t)]Kx(t) \tag{3}$$

where, K is the nominal controller gain, α is the positive constant, and the term $\alpha\Phi(t)K$ represents controller gain variations and $\Phi(t)$ is defined as

$$\Phi(t)^T \Phi(t) \le I. \tag{4}$$

Here, the value of α indicates the measure non-fragility against controller gain variations. Now, the closed loop system from the singular system (1) and the controller (3) is given by

$$\begin{aligned} E\dot{x}(t) &= [A + B_1(I + \alpha\Phi(t))K]x(t) + B_2 w(t) \\ z(t) &= Cx(t). \end{aligned} \tag{5}$$

Also, we introduce H_∞ performance measure

$$\int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)]dt \tag{6}$$

The aim of this paper is that the closed loop system (5) is asymptotically stable with disturbance attenuation γ and non-fragility α. Therefore, the objective is designing a non-fragile H_∞ state feedback controller K in the presence of disturbance input in the singular system and the multiplicative uncertainty of the controller. Here, we summarize some definitions and useful properties [1] for descriptor systems in the following.

Definition 1. For the descriptor system $E\dot{x}(t) = Ax(t)$, if *det(sE-A)* is not identically zero, a pencil *sE-A* (or a pair (*E,A*)) is regular. The property of regularity guarantees the existence and uniqueness of solution for any specified initial condition. The singular system has no impulsive mode (or impulse free) if and only if *rank(E)=degdet(sE-A)*. The condition of impulse free ensures that singular system has no infinite poles.

Theorem 1. Consider a closed loop singular system (5). If there exist invertible symmetric matrix P and feedback gain K satisfying

$$E^T P = PE \ge 0 \tag{7}$$

$$\begin{bmatrix} \Pi & P^T B_2 \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{8}$$

then, the controller (2) is a non-fragile H_∞ controller guaranteeing the asymptotic stability in the presence of controller gain variations and H_∞ norm bound of the closed loop singular systems. Here, Π is defined as follows:

$$\begin{aligned} \Pi = {} & A^T P + P^T A + C^T C + P^T B_1 K + K^T B_1^T P + \alpha\varepsilon P^T B_1 B_1^T P \\ & + \alpha\varepsilon P^T B_1 B_1^T P + \frac{\alpha}{\varepsilon} K^T K. \end{aligned} \tag{9}$$

Proof. For asymptotic stability of closed loop singular system (5), if we take a Lyapunov functional

$$V(x(t)) = x(t)^T E^T P x(t) \tag{10}$$

with $E^T P = PE \ge 0$, then the time derivative of (10) is given by

$$\dot{V}(x(t)) = \dot{x}(t)^T E^T P x(t) + x(t)^T P^T E\dot{x}(t). \tag{11}$$

The matrix inequality (8) from the Lyapunov functional (10) and H_∞ performance measure (6) implies

$$z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \dot{V}(x(t)) < 0. \tag{12}$$

Therefore, using the following lemma

$$\begin{aligned} & 2\alpha x(t)^T P B_1 \Phi(t) K x(t) \\ & \le \alpha\varepsilon x(t)^T P B_1 B_1^T P x(t) + \frac{\alpha}{\varepsilon} x(t)^T K^T K x(t), \end{aligned} \tag{13}$$

the inequality (12) is equal to

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \Pi & P^T B_2 \\ * & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < 0. \tag{14}$$

Hence, the inequality (14) implies the sufficient condition (8).

However, it is not easy to solve Theorem 1, because the sufficient condition of (8) is not an LMI from and the equality condition is included in (7). In order to make a perfect LMI condition in terms of finding all variables and eliminate equality condition, the obtained sufficient conditions are changed in the following Theorem 2 by proper manipulations. Moreover, the non-fragile H_∞ controller design method for singular systems is presented.

Theorem 2. If there exist positive definite matrix Q_1, an invertible symmetric matrix Q_4, matrices Q_3, Y_1, Y_2, and positive scalar β_1, β_2, ρ satisfying

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & B_{21} & Q_1 C_1^T + Q_3^T C_2^T & Q_1 & Q_3^T & Y_1^T \\ * & \Gamma_3 & B_{22} & Q_4 C_2^T & 0 & Q_4 & Y_2^T \\ * & * & -\rho I & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -\beta_2 I \end{bmatrix} < 0 \tag{13}$$

then the matrix expressed by

$$K = \begin{bmatrix} Y_1 P_1 + Y_2 P_3 & Y_2 P_4 \end{bmatrix} \tag{14}$$

is a controller gain in non-fragile H_∞ controller satisfying asymptotic stability, regular, impulse-free, and H_∞ norm bound in the presence of controller gain variations and disturbance input. Here, some notations are defined as follows:

$$\begin{aligned} \Gamma_1 &= A_1 Q_1 + Q_1 A_1^T + B_{11} Y_1 + Y_1^T B_{11}^T + \beta_1 B_{11} B_{11}^T \\ \Gamma_2 &= Q_3^T A_4^T + B_{11} Y_2 + Y_1^T B_{12}^T + \beta_1 B_{11} B_{12}^T \\ \Gamma_3 &= A_4 Q_4 + Q_4 A_4^T + B_{12} Y_2 + Y_2^T B_{12}^T + \beta_1 B_{12} B_{12}^T \\ P_1 &= Q_1^{-1} \\ P_4 &= Q_4^{-1} \end{aligned}$$

$$P_3 = -P_4 Q_3 P_1$$
$$\beta_1 = \alpha\varepsilon$$
$$\beta_2 = \frac{\varepsilon}{\alpha}$$
$$\rho = \gamma^2.$$

Proof. Using Schur complements and some changes of variables, the matrix inequality (8) is transformed into

$$\begin{bmatrix} \Psi & B_2 \\ * & -\rho I \end{bmatrix} < 0 \tag{15}$$

where, Ψ is defined

$$\begin{aligned} \Psi &= Q^T A^T + AQ + Q^T C^T CQ + Q^T Q + B_1 KQ \\ &\quad + Q^T K^T B_1^{\ T} + \alpha\varepsilon B_1 B_1^{\ T} + \frac{\alpha}{\varepsilon} Q^T K^T KQ. \end{aligned} \tag{16}$$

To obtain an LMI sufficient condition in terms of finding all variables and eliminate the equality in (7), we make use of singular value decomposition and changes of variables. Without loss of generality, we assume that the system matrices of (1) have the following singular decomposition form [7]

$$\begin{aligned} E &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \\ B_2 &= \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{aligned} \tag{17}$$

where all matrices have appropriate dimensions. Also, if we set

$$P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix} \tag{18}$$

In order to satisfy (7), if other solutions have the following structure

$$\begin{aligned} Y &= \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \\ Q &= P^{-1} = \begin{bmatrix} Q_1 & 0 \\ Q_3 & Q_4 \end{bmatrix} \end{aligned} \tag{19}$$

And we apply (17), (18), and (19) to (15), then (15) is changed to (13). Therefore, (13) is an LMI form in terms of all variables, Q_1, Q_3, Q_4, Y_1, Y_2, β_1, β_2, and ρ.

Remark 1. In the case of $E = I$, the problem can be solved directly from an LMI condition in (15). Therefore, the proposed design algorithm can be solved in non-singular systems with an LMI condition. Thus, the result is general design method.

Remark 2. (13) is an LMI form in terms of all variables, Q_1, Q_3, Q_4, Y_1, Y_2, β_1, β_2, and ρ. Therefore, non-fragile H_∞ state feedback controller can be calculated directly using LMI Toolbox. Also, the measure of non-fragility in controller, α, can be calculated by $\alpha = \sqrt{\beta_1/\beta_2}$ and the value of disturbance attenuation, $\gamma = \sqrt{\rho}$.

Remark 3. In order to get a minimum value of γ or maximum value of α, the LMI feasibility problem in Theorem 2 can be reformulated. To obtain the minimum value of γ, the optimization problem is rewritten as

Maximize ρ subject to LMI (13).

In the case of maximum value of α, the optimization problem is modified by

Maximize β_1 subject to LMI (13).
Minimize β_2 subject to LMI (13).

However, it is difficult to obtain the value of minimum value of γ and maximum value of α at the same time. Therefore, one of the future researches is to develop synthesis algorithms which take into account certain structured uncertainties in the controllers and search for the best solution that guarantees a compromise between optimality and fragility.

The proposed non-fragile H_∞ controller design algorithm can be extended into robust and non-fragile H_∞ control problem for singular systems with parameter uncertainties in the following Corollary 1.

Corollary 1. Consider a parameter uncertain singular system

$$\begin{aligned} E\dot{x}(t) &= [A + \Delta A(t)]x(t) + [B_1 + \Delta B_1(t)]u(t) + [B_2 + \Delta B_2(t)]w(t) \\ z(t) &= Cx(t) \end{aligned} \tag{20}$$

with parameter uncertainties

$$\begin{aligned} &[\Delta A(t) \quad \Delta B_1(t) \quad \Delta B_2(t)] = HF(t)[E_1 \quad E_2 \quad E_3] \\ &F(t)^T F(t) \le I \end{aligned} \tag{21}$$

where, H and E_i $(i = 1,2,3)$ are known matrices. This system (20) is stabilizable and has an H_∞ performance $\gamma > 0$ by a non-fragile H_∞ state feedback controller if and only if there exist a $\lambda > 0$ such that singular systems without parameter uncertainties.

$$\begin{aligned} E\dot{x}(t) &= A(x(t) + B_1 u(t) + \begin{bmatrix} B_2 & \gamma\lambda H \end{bmatrix} \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} &= \begin{bmatrix} C \\ \frac{1}{\lambda} E_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{\lambda} E_2 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ \frac{1}{\lambda} & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} \end{aligned} \tag{22}$$

with additional disturbance input variable $\hat{w}(t)$ and additional controlled output variable $\hat{z}(t)$, is stabilizable and has an non-fragile H_∞ performance $\gamma > 0$ by the same state feedback controller.

Proof. Using the existing results, it is easily proved.

Therefore, the problem of robust and non-fragile H_∞ control for uncertain singular systems can be solvable using the presented method.

To demonstrate the validity of the proposed method, a singular system is considered as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}w(t) \qquad (23)$$
$$z(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}x(t).$$

All solutions satisfying Theorem 2 are obtained at the same time using LMI Toolbox [10] as follows:

$$Q = \begin{bmatrix} 0.3068 & -0.1174 & 0 \\ -0.1174 & 0.0687 & 0 \\ 0.0425 & 0.2919 & 0.0778 \end{bmatrix}$$
$$Y = \begin{bmatrix} -0.0450 & -0.6029 & -0.5098 \end{bmatrix}$$
$$P = \begin{bmatrix} 9.4328 & 16.1289 & 0 \\ 16.1289 & 42.1378 & 0 \\ -65.6772 & -166.9290 & 12.8539 \end{bmatrix} \qquad (24)$$
$$\beta_1 = 70.3712$$
$$\beta_2 = 1.1905$$
$$\rho = 0.9266$$

Therefore, non-fragile H_∞ state feedback gain, H_∞ norm bound, and the value of non-fragility in controller are obtained from (14) and Remark 2 as follows:

$$K = \begin{bmatrix} 23.2315 & 58.7034 & -6.5528 \end{bmatrix}$$
$$\alpha = \sqrt{\beta_1/\beta_2} = 0.5584$$
$$\gamma = \sqrt{\rho} = 0.9626$$

The meaning of α implies that the obtained non-fragile H_∞ controller guarantees asymptotic stability and disturbance attenuation, $\| z(t) \|_2 \le 0.9626 \| w(t) \|_2$ for any $w(t) \in L_2[0,\infty)$ of the closed loop system in spite of controller gain variations within 55.84%.

3. CONCLUSION

In this paper, we treated the problem of non-fragile H_∞ state feedback controller design algorithm for singular systems and static state feedback controller with multiplicative uncertainty. The sufficient condition and non-fragile H_∞ controller design method was discussed. Moreover, it was shown that the presented design algorithm can be directly applied to the singular systems with parameter uncertainties using proper manipulations.

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