Design of observer for a class of Lipschitz nonlinear systems

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Abstract: The problem of observer design for a class of Lipschitz nonlinear systems is considered. We propose a new observer design method which takes into account the structure of perturbed nonlinear terms with a scaling factor ϵ. An example is provided to demonstrate the usefulness of our result over the existing method.

Keywords: Nonlinear observer design; exponential stability

1. Introduction

The observer design for nonlinear systems has received much attention [1]-[5]. In particular, for Lipschitz nonlinear systems, the optimal design method of the observer gain L is proposed [4] for the purpose of maximizing the allowable Lipschitz constant (denoted as γ) imposed on the perturbed nonlinear terms. This result is later extended to systems where the reduced-order observer exists with the same bound on the Lipschitz constant [5]. However, the obtained bound on γ is usually small because they try to maximize $\gamma < 1/(2\lambda_{\max}(P))$ where the positive definite matrix P is a unique solution of the Lyapunov equation [4]-[5].

In this paper, we propose a new observer design method which takes into account the structure of perturbed nonlinear terms with a scaling factor ϵ. We show that for a certain class of Lipschitz nonlinear systems, the conservative bound on γ can be much relaxed. Moreover, we show that the scaling factor ϵ can be properly tuned from low-gain to high-gain depending on the nature of the perturbed terms. This is a different result from [1]-[3] where only the design of high-gain observer is addressed.

Throughout the paper, the Euclidean 2-norm is used. Otherwise, it will be specifically denoted by subscript.

2. Review

In this section, we review the recent result on designing the observer for Lipschitz nonlinear systems [4] The system under consideration in [4] is

$$\begin{aligned} \dot{x} &= Ax + \Phi(x,u) \\ y &= Cx \end{aligned} \qquad (1)$$

where $x \in \mathbf{R^n}$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the state, the input, and the output respectively. Moreover, the system satisfies that (i) (A, C) is observable; (ii) $\|\Phi(x,u) - \Phi(\hat{x},u)\| \le \gamma\|x - \hat{x}\|$. The observer is in the form of

$$\dot{\hat{x}} = A\hat{x} + \Phi(\hat{x},u) + L(y - C\hat{x}) \qquad (2)$$

Theorem 1: [4] For (1), the observer given by (2) is asymptotically stable if L is chosen such that (i) $A - LC$ is stable; (ii) $\min_{w\in\mathbf{R}^+} \sigma_{\min}(A - LC - jwI) > \gamma$.

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Observation: Theorem 1 provides the sufficient condition on the Lipschitz constant γ, which guarantees the asymptotic stability of the observer for a general Lipschitz nonlinear system (1). However, the obtained bound on γ by Theorem 1 is usually small (i.e. $\gamma < 1$). In the following, we show that for a certain class of Lipschitz nonlinear systems, the conservativeness in obtaining the bound on γ can be much relaxed.

3. Main Results

We consider a class of Lipschitz nonlinear systems given by

$$\begin{aligned} \dot{x} &= Ax + \Phi(x,u) \\ y &= Cx \end{aligned} \qquad (3)$$

where $x \in \mathbf{R^n}$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the state, the input, and the output, respectively. Moreover, the perturbed nonlinear term is $\Phi(x,u) = [\phi_1(x,u), \cdots, \phi_n(x,u)]^T \in \mathbf{R^n}$. The system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

For a nonlinear term $\Phi(x,u)$, we assume that there exists a function $\gamma(\epsilon) \ge 0$ such that for any $\epsilon > 0$

$$\|E(\epsilon)\left(\Phi(x,u) - \Phi(\hat{x},u)\right)\| \le \gamma(\epsilon)\|E(\epsilon)\left(x - \hat{x}\right)\| \qquad (4)$$

where $E(\epsilon) = \mathrm{diag}[1, \epsilon, \cdots, \epsilon^{n-1}]$.

Our nonlinear observer is

$$\dot{\hat{x}} = A\hat{x} + L(\epsilon)(y - C\hat{x}) + \Phi(\hat{x},u) \qquad (5)$$

where $L(\epsilon) = [\frac{l_1}{\epsilon}, \cdots, \frac{l_n}{\epsilon^n}]^T$ with $\epsilon > 0$.

Observer design procedure:

(i) Obtain a function $\gamma(\epsilon)$ that satisfies (4).

(ii) Select $L = [l_1, \cdots, l_n]^T$ such that $A_L := A - LC$ is Hurwitz.

(iii) Obtain the solution P of the Lyapunov equation $A_L^T P + PA_L = -I$.

(iv) Select ϵ such that $\epsilon^{-1} - \rho\gamma(\epsilon) > 0$ where $\rho = 2\|P\|$.

Theorem 2: For (3), the observer given by (5) is exponentially stable if there exist L and ϵ such that A_L is Hurwitz and $\epsilon^{-1} - \rho\gamma(\epsilon) > 0$.

Proof: Define $e := x - \hat{x}$. By subtracting (5) from (3), the error dynamics is

$$\dot{e} \;=\; A_L(\epsilon)e + \Phi(x,u) - \Phi(\hat{x},u) \qquad (6)$$

where $A_L(\epsilon) = A - L(\epsilon)C$.

Since A_L is Hurwitz, we have a Lyapunov equation $A_L^T P + PA_L = -I$. With the matrix $E(\epsilon)$ as defined above, we have the following equalities: $\epsilon A_L(\epsilon) = E(\epsilon)^{-1}A_L E(\epsilon)$, $A_L^T(\epsilon)P(\epsilon) + P(\epsilon)A_L(\epsilon) = -\epsilon^{-1}E^2(\epsilon)$, and $P(\epsilon) = E(\epsilon)PE(\epsilon)$. Now we set a Lyapunov function $V(e) = e^T P(\epsilon)e$. Then, along the trajectory of (6),

$$\begin{aligned}\dot{V}(e) &= -\epsilon^{-1}\|E(\epsilon)e\|^2 + 2e^T E(\epsilon)PE(\epsilon)\left(\Phi(x,u) - \Phi(\hat{x},u)\right) \\ &\le -\epsilon^{-1}\|E(\epsilon)e\|^2 \\ &\quad +2\|P\|\|E(\epsilon)e\|\|E(\epsilon)\left(\Phi(x,u) - \Phi(\hat{x},u)\right)\| \qquad (7)\end{aligned}$$

Here, from (4), we have $\|E(\epsilon)\left(\Phi(x,u) - \Phi(\hat{x},u)\right)\| \le \gamma(\epsilon)\|E(\epsilon)e\|$. This leads to

$$\dot{V}(e) \le -N\|E(\epsilon)e\|^2, \quad N = \epsilon^{-1} - \rho\gamma(\epsilon) > 0 \qquad (8)$$

Also, note that

$$\lambda_{\min}(P)\|E(\epsilon)e\|^2 \le V(e) \le \lambda_{\max}(P)\|E(\epsilon)e\|^2 \qquad (9)$$

From (8) and (9),

$$V(e) \le V(e(0))e^{-\frac{N}{\lambda_{\max}(P)}t} \qquad (10)$$

Then, using (9) and (10),

$$\lambda_{\min}(P)\|E(\epsilon)e\|^2 \le V(e) \le \lambda_{\max}(P)\|E(\epsilon)e(0)\|^2 e^{-\frac{N}{\lambda_{\max}(P)}t} \quad (11)$$

which leads to

$$\|E(\epsilon)e\| \le \|E(\epsilon)e(0)\|\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}e^{-\frac{N}{2\lambda_{\max}(P)}t} \qquad (12)$$

Thus, the state e converges to zero exponentially. □

The significance of Theorem 2 over Theorem 1 becomes clear if we consider the following two cases which represent the models of several practical systems [3]:

(a) (Triangular-Type Lipschitz Term): For $i = 1, \cdots, n$, there exists a constant $\gamma \ge 0$ such that $|\phi_i(x,u) - \phi_i(\hat{x},u)| \le \gamma(|x_1 - \hat{x}_1| + \cdots + |x_i - \hat{x}_i|)$.

(b) (Feedforward-Type Lipshitz Term): For $i = 1, \cdots, n-2$, there exists a constant $\gamma \ge 0$ such that $|\phi_i(x,u) - \phi_i(\hat{x},u)| \le \gamma(|x_{i+2} - \hat{x}_{i+2}| + \cdots + |x_n - \hat{x}_n|)$ with $\phi_{n-1}(x,u) = \phi_n(x,u) = 0$.

Theorem 3: Suppose that either the condition (a) or (b) is satisfied. Then, for (3), the observer given by (5) is exponentially stable for any finite constant γ.

Proof: Since A_L can be made Hurwitz by L, we only need to show that there exist some ϵ such that $\epsilon^{-1} - \rho\gamma(\epsilon) > 0$. Note that for vectors $X \in \mathbf{R^n}$ and $Y \in \mathbf{R^n}$, if $\|X\|_1 \le \gamma\|Y\|_1$ holds, then $\|X\| \le \sqrt{n}\gamma\|Y\|$ holds. With this property, under the condition (a), it is easy to obtain that $\gamma(\epsilon) = \sqrt{n}\gamma(1 + \epsilon + \cdots + \epsilon^{n-1})$. Thus, for any finite constant γ, there exists ϵ^* such that $\epsilon^{-1} - \rho\gamma(\epsilon) > 0$ for $0 < \epsilon < \epsilon^*$. Similarly, under the condition (b), it is easy to obtain that $\gamma(\epsilon) = \sqrt{n}\gamma(\epsilon^{-2} + \cdots + \epsilon^{-(n-1)})$. Thus, for any finite constant γ, there exists ϵ^* such that $\epsilon^{-1} - \rho\gamma(\epsilon) > 0$ for $\epsilon^* < \epsilon < \infty$. □

Under the condition (a), the proposed observer becomes a high-gain observer as studied in [1]-[3]. However, under the condition (b), it becomes a 'low-gain' observer, which has not been addressed in [1]-[3].

4. Illustrative example

The example used in [4] is $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. In [4], the obtained bound on γ is 0.49. If $\Phi(x,u) = [-10\mathrm{sat}(x_1), x_2 \sin u]^T$, the observer design method in [4] is not applicable. To apply our method, we rename the states such as $z_1 = x_2$ and $z_2 = x_1$. Then, we have

$$\begin{aligned}\dot{z} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} z_1 \sin u - z_1 \\ -10\mathrm{sat}(z_2) \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z \qquad (13)\end{aligned}$$

Now, we design a nonlinear observer by the proposed method.

(i) In view of (4), we have $|\phi_1(z,u) - \phi_1(\hat{z},u)| \le |z_1 - \hat{z}_1|$ and $|\phi_2(z,u) - \phi_2(\hat{z},u)| \le 10|z_2 - \hat{z}_2|$. Thus, we obtain $\gamma(\epsilon) = 10\sqrt{2} = 14.1421$.

(ii) Select $L = [-4, -4]^T$ (all eigenvalues of A_L at -2).

(iii) The solution of Lyapunov equation is $P = \begin{bmatrix} 0.1562 & 0.1250 \\ 0.1250 & 1.1250 \end{bmatrix}$.

(iv) We have $\rho = 2\|P\| = 2.2817$. Thus, the range of ϵ is $0 < \epsilon < \epsilon^* := 1/(\rho\gamma(\epsilon)) = 0.0310$. We select $\epsilon = 0.03$.

The simulation is performed with $u = 0$. The initial conditions are set as $x_1(0) = 2$, $x_2(0) = 0.1$, $\hat{x}_1(0) = 0$, and $\hat{x}_2(0) = 0$. In Fig. 1, it is shown that the proposed observer dominates the nonlinear terms and achieves the exponential tracking of the real state.

5. Conclusions

We have presented the new result on designing an observer for a class of Lipschitz nonlinear systems. Utilizing the scaling factor coupled with the structure of Lipschitz nonlinear terms, we obtain a much relaxed bound on the Lipschitz constant γ over the existing method.

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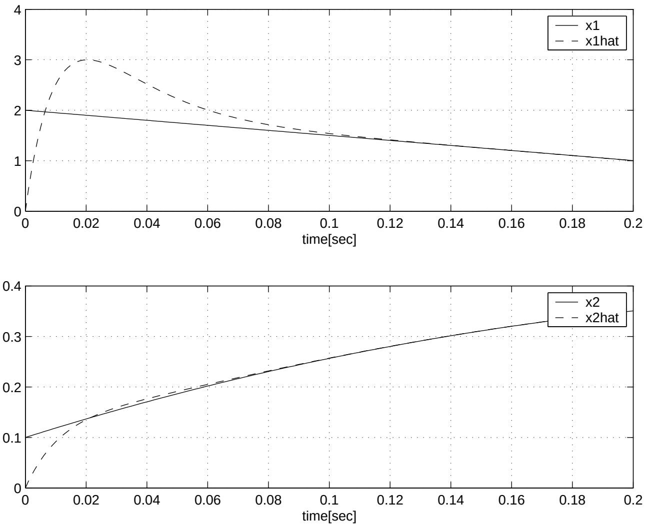

Fig. 1. Observer performance: real state and its estimate.