Optimal H_2 design of the one-degree-of-freedom decoupling controllers

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Abstract : In this paper, H_2 designs for the one-degree-of-freedom decoupling control systems are treated for the generalized plant. The optimal H_2 controller is obtained together with the ones that yield finite H_2 cost functions under compact assumptions. It is shown that the optimal closed transfer matrix is strictly proper under the reasonable order assumptions on the generalized plant.

Keyword : Decoupling controller, H_2 design, General plant model, One-degree-of-freedom controller.

1. Introduction

Decoupling control system design that eliminates coupling interactions between the various reference and manipulated signals has been the interest of many researchers. Desoer and Gundes (1986), Lee and Bongiorno (1993) and Park et. al.(2002) solve the two-degree-of-freedom decoupling problems. The decoupling problem for the one-degree-of-freedom control system is treated by Gomez and Goodwin (2000), Youla and Bongiorno (2000) and Bongiorno and Youla(2001). Gomez and Goodwin(2000) treat both partial and diagonal decoupling designs by adopting an algebraic approach based on coprime factorizations. A notable feature of the work by Youla and Bongiorno (2000) is that the solvability condition and the characterization of the all decoupling closed loop transfer matrices are explicitly expressed in the most effective way. They also consider the optimal designs for the persistent inputs which include the most practical signals such as step or ramp functions. In this paper, the work of Youla and Bongiorno(2000) is extended to the generalized plant.

Throughout the paper, only real rational matrices are considered. The notation $\| T \|$ denotes the L_2 or H_2 norm of the transfer matrix $T(s)$. A rational matrix $G(s)$ is called stable if it is analytic in $Re\, s \geq 0$. The notations G^T and $\det G$ are used for the transpose and determinant of G, respectively. The notation $G_*(s)$ stands for $G^T(-s)$. In the partial fraction expression of $G(s)$, the contribution made by all its finite poles in $Re\, s \leq 0, Re\, s > 0$ and by its pole at $s = \infty$ are denoted by $\{G\}_+, \{G\}_-$ and $\{G\}_\infty$, respectively. The order relationship $G(s) \leq O(s^k)$ means that no entry in $G(s)$ grows faster than s^k as $s \to \infty$. A diagonal matrix G with g_i in the $i-$row, $i-$column is denoted by $G = diag\{g_1, g_2, \cdots, g_n\}$ or simply by $diag\{g_i\}$. The Schur product of two matrices is denoted as $G \circ R$ and is the matrix whose $i-$row, $j-$column is given by $g_{ij} r_{ij}$. The Kronecker product of two matrices is denoted as $G \otimes R$ and is the matrix whose $ij-$block is given by $g_{ij} R$. The vector $vec\, G = \left[g_1^T, g_2^T, \cdots, g_n^T \right]^T$ is formed by stacking all the columns of the matrix G. The Khatri-Rao product of two matrices is denoted as $G \odot R$ and is the matrix whose i-column is given by $g_i \otimes r_i$ where g_i and r_i the i-column of G and i-column of R, respectively. For a diagonal matrix

$G = diag\{g_1, g_2, \cdots, g_n\}$, the vector $vecd\,G = [g_1\ g_2\ \cdots\ g_n]^T$ is formed by stacking all the diagonal elements of the matrix G. When V is a diagonal matrix, $vec(AVD) = (D^T \otimes A)\,vec\,V = (D^T \odot A)\,vecd\,V$ (Brewer, 1978). A rational matrix $G(s)$ is said to be biproper if both $G(s)$ and $G^{-1}(s)$ are proper.

2. Formulation of the decoupling control problems

The plant model under consideration is given in Figure 1.

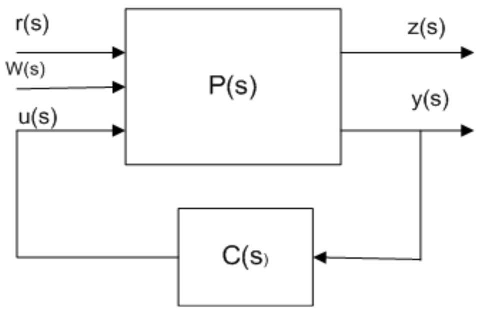

Figure 1. The general plant model

The exogenous inputs are divided into two variables of the reference input $r(s)$ and the other exogenous inputs $w(s)$. The variables $u(s)$, $z(s)$ and $y(s)$ are the plant input, the regulated variable and the controller input, respectively and the transfer matrix of the general plant is given by

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} r \\ w \\ u \end{bmatrix}, \text{ with } P = \begin{bmatrix} P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \quad (1)$$

When the plant satisfies the following assumption, called admissibility condition, a stabilizing controller exists(Park and Bongiorno,1989):

Assumption 1: The general plant block $P(s)$ is free of hidden modes in $Re\ s \geq 0$ and $\Psi_P^+ = \Psi_{p_{22}}^+$.

The notation Ψ_P denotes the characteristic denominator(Youla et al, 1976) of the rational matrix $P(s)$ and Ψ_P^+ absorbs all the zeros in $Re\ s \geq 0$.

Decoupling design is to make the transfer matrix from the reference input to the plant output diagonal and invertible. In the general plant model, however, the plant output does not explicitly appear and hence, careful consideration is needed to formulate a meaningful decoupling problem. For 1DOF controller configuration with unity feedback, the controller input is usually the difference between the reference input $r(s)$ and the plant output, say $\hat{y}(s)$, possibly with sensing noises and disturbances added. Since the controller input is embodied by $y(s)$ in the general plant, it can be written as

$$y = r - \hat{y} = r - T_{\hat{y}r} r - T_{\hat{y}w} w = (I - T_{\hat{y}r}) r - T_{\hat{y}w} w \quad (2)$$

Since $y = T_{yr} r + T_{yw} w$, we conclude that $T_{yr} = I - T_{\hat{y}r}$. The target transfer matrix to be diagonalized is $T_{\hat{y}r}$ and hence $T_{\hat{y}r} = I - T_{yr} = I - (I - P_{22})^{-1} P_{20}$ is the target matrix. Next, we make an assumption that can characterize 1DOF controller configuration. From (1), we see that $y(s) = P_{20} r + P_{21} w + P_{22} u$. Comparing this equation to (2), we can match the variables as $r = P_{20} r$ and $\hat{y} = -P_{21} w - P_{22} u$ and this leads the assumption that $P_{20} = I$. Hence the target matrix is $T := I - (I - P_{22} C)^{-1} = -P_{22} C (I - P_{22} C)^{-1}$. In summary, we formulate the decoupling control problem for 1DOF controllers as the one of finding the stabilizing controllers that make the transfer matrix

$$T(s) = -P_{22} C (I - P_{22} C)^{-1} \quad (3)$$

diagonal and invertible with the following assumption;

Assumption 2: The matrix $P_{22}(s)$ is square and invertible, and $P_{20} = I$.

A rational matrix $T(s)$ is said to be realizable for the plant $P_{22}(s)$ if the corresponding controller $C(s)$ in (3) stabilizes the feedback system in Figure 1. A controller $C(s)$ is said to be decoupling for the plant $P_{22}(s)$ if it stabilizes the loop and produces a diagonal $T(s)$ with $\det T(s) \neq 0$.

Lemma 1: A rational matrix $T(s)$ is realizable if and only if the four matrices T, $P_{22}^{-1}T$, $P_{22}^{-1}TP_{22}$ and $(I-T)P_{22}$ are stable with $\det(I-T) \neq 0$.

The following lemma describes the existence condition of a decoupling controller and the proof of the lemma can be seen in Youla and Bongiorno (2000). Let us first define two polynomials to describe the decoupling condition. Let the $i-th$ column of $P_{22}^{-1}(s)$ and the $i-th$ row of $P_{22}(s)$ be denoted by $PI_{ci}(s)$ and $P_{ri}(s)$, respectively. Let $\theta_i(s)$ and $\psi_i(s)$ denote the unique monic polynomials of the minimal degrees such that $PI_{ci}(s)\theta_i(s)$ and $\psi_i(s)P_{ri}(s)$ are stable, respectively.

Lemma 2 (Youla and Bongiorno, 2000): A decoupling controller for the plant $P_{22}(s)$ exists if and only if 1) the polynomials $\theta_i(s)$ and $\psi_i(s)$ are coprime for $i=1\to n$ and 2) the unique data construct $\Sigma(s)=P_{22}^{-1}\Delta_\theta\Delta_\alpha P_{22}$ is stable where $\Delta_\theta = diag\{\theta_i\}$, $\Delta_\alpha = diag\{\alpha_i\}$ and the polynomial $\alpha_i(s)$ is such that

$$\alpha_i\theta_i + \beta_i\psi_i = 1, \text{ for } i=1\to n \tag{4}$$

The existence of such α_i and β_i is always guaranteed if θ_i and ψ_i are coprime. When a decoupling controller exists, any realizable diagonal transfer matrix $T(s)$ is given by

$$T(s)=\Delta_\theta(\Delta_\alpha + \Delta\,\Delta_\psi) \tag{5}$$

where $\Delta_\psi = diag\{\psi_i\}$ and $\Delta(s)$ is an arbitrary stable diagonal matrix chosen so that $\det(I-T(s)) \neq 0$.

3. H_2 optimal design problem

In this section, we formulate H_2 design for the decoupling controllers. To allow the persistent exogenous inputs, we assume that the reference input $r(s)$ and exogenous input $w(s)$ are generated by

$$r(s)=P_r(s)\,r_o \text{ and } w(s)=P_w(s)\,w_o, \tag{6}$$

where the vectors r_o and w_o behave in impulsive forms. Inserting (6) into (1), we obtain

$$\begin{bmatrix} z \\ y \end{bmatrix} = \tilde{P}\begin{bmatrix} r_o \\ w_o \\ u \end{bmatrix}, \text{ with } \tilde{P} = \begin{bmatrix} P_{10}P_r & P_{11}P_w & P_{12} \\ P_{20}P_r & P_{21}P_w & P_{22} \end{bmatrix} \tag{7}$$

Define $\tilde{P}_{11} = [\,P_{10}P_r \;\; P_{11}P_w\,]$ and $\tilde{P}_{21} = [\,P_r \;\; P_{21}P_w\,]$ so that $\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & P_{12} \\ \tilde{P}_{21} & P_{22} \end{bmatrix}$. Let the transfer matrix from $[r_o^T \;\; w_o^T]^T$ to z be denoted by $T_{zo}(s)$. Our problem is to find the decoupling controller in (5) that minimizes $\| T_{zo} \|$. Since we deal with the 1DOF controller, we assume the following one whose explanation is well presented in Park (2005);

Assumption 3: $\Psi_{\tilde{P}}^{+} = \Psi_{P_{22}}^{+}$

The following assumptions guarantee that the optimal decoupling controller exists.

Assumption 4: $P_{10}P_r$ and $P_{11}P_w$ are strictly proper.

Assumption 5: $\det(T_{a*}(s)T_a(s)) \neq 0$ on the finite part of the $s=j\omega-axis$, where $T_a(s)=P_{12}(s)P_{22}^{-1}(s)\Delta_\theta(s)$.

Assumption 6: $\det(T_b(s)T_{b*}(s)) \neq 0$ on the finite part of the $s=j\omega-axis$ where $T_b(s)=\Delta_\psi(s)\tilde{P}_{21}(s)$.

Assumption 7: The order relationships $(P_{12*}P_{12})^{-1} \leq O(s^{2k_1})$, $(\tilde{P}_{21}\tilde{P}_{21*})^{-1} \leq O(s^{2k_2})$ and $P_{22} \leq O(s^{k_3})$ are satisfied with $k_1+k_2+k_3 \leq 0$.

Let $\Omega(s)$ be the Wiener-Hopf spectral factor of the equation

$$\left(T_b^T \odot T_a\right)_*(T_b^T \odot T_a) = (T_bT_{b*})^T \circ (T_{a*}T_a)$$

$$= \Omega_*(s)\,\Omega(s) \qquad (8)$$

and define

$$U(s) = (T_b^T \odot T_a)\,\Omega^{-1} \text{ and}$$
$$T_{11} = \tilde{P}_{11} - P_{12}P_{22}^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21} \qquad (9)$$

For later use we present the following lemmas whose proofs can be seen in Appendix.

Lemma 3: Suppose that Assumptions 1-3 are satisfied and the plant $P_{22}(s)$ admits a decoupling controller (Lemma 2). The matrices T_{11}, T_a and T_b are stable.

Remark: When the plant $P_{22}(s)$ admits a decoupling controller and Assumptions 1-6 are met, Ω and Ω^{-1} are stable and U is inner.

Lemma 4: Let Q and R be $l\times l$ rational matrices and suppose that the inverses Q^{-1}, R^{-1} and $(Q\circ R)^{-1}$ exist. If $Q^{-1}\le O(s^{k_1})$ and $R^{-1}\le O(s^{k_2})$, then $(Q\circ R)^{-1}\le O(s^{k_1+k_2})$.

Now we present the main results on the H_2 problem.

Theorem: Suppose that the general plant is admissible (Assumption 1) and the plant $P_{22}(s)$ admits a decoupling controller (Lemma 2). Under Assumptions 2~7, the class of all decoupling closed transfer matrices that yield finite cost $\|T_{zo}\|$ is given by

$$vecd\, T(s) = \Delta_\theta\Delta_\psi\Omega^{-1}(\{b\}_+ + \{c\}_- + \delta_f) \qquad (10)$$

where $b = U_*\, vec\,\tilde{P}_{11}$, $c = \Omega\, vecd(\Delta_\alpha\Delta_\psi^{-1})$ and $\delta_f(s)$ is an arbitrary stable vector $\le O(s^{-1})$. In this case, $T(s)$ in (10) is strictly proper and the corresponding controller $C(s)$ can be obtained form (3). The optimal controller is the one with $\delta_f = 0$ and the cost E for the controller (10) is given by $E = \tilde{E} + \|\delta_f\|^2 \ge \tilde{E}$, where $\tilde{E}$ denotes the cost for the optimal controller.

Proof: Notice that $T_{zo} = \tilde{P}_{11} - P_{12}C(I-P_{22}C)^{-1}\tilde{P}_{21} = \tilde{P}_{11} - P_{12}P_{22}^{-1}T\tilde{P}_{21}$ and inserting the realizable formula for $T(s)$ in (5) into the above equation yields $T_{zo} = \tilde{P}_{11} - P_{12}P_{22}^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21} - P_{12}P_{22}^{-1}\Delta_\theta\,\Delta\,\Delta_\psi\tilde{P}_{21} = T_{11} - T_a\,\Delta\,T_b$. Since the vectorization does not affect the H_2 norm, $J = \|T_{zo}\|^2 = \|vec\,T_{zo}\|^2 = \|vec\,T_{11} - (T_b^T\odot T_a)\,vecd\,\Delta\|^2$. Since U is inner, there exists a complementary inner $U_\perp$ such that $[\,U \,\vdots\, U_\perp\,]$ is square inner. Notice that $[\,U \,\vdots\, U_\perp\,]$ is stable and proper. It now follows that

$$\begin{aligned} J &= \left\| vec\,T_{11} - [U\;\; U_\perp]\begin{bmatrix}\Omega\; vecd\,\Delta\\ 0\end{bmatrix}\right\|^2 \\ &= \left\| \begin{bmatrix}U_*\\ U_{\perp *}\end{bmatrix} vec\,T_{11} - \begin{bmatrix}\Omega\; vecd\,\Delta\\ 0\end{bmatrix}\right\|^2 \\ &= \left\| \begin{bmatrix}U_*\, vec\,T_{11} - \Omega\, vecd\,\Delta\\ U_{\perp *}\, vec\,T_{11}\end{bmatrix}\right\|^2 \end{aligned} \qquad (11)$$

Hence, minimization is achieved by taking $vecd\,\Delta_{opt} = \Omega^{-1}(\{U_*\, vec\,T_{11}\}_+ + \{U_*\, vec\,T_{11}\}_\infty)$. We will show that $vecd\,\Delta_{opt}$ is stable and the cost function in (11) is finite. In view of Lemma 3, $vecd\,\Delta_{opt}$ is obviously stable. The H_2 norm of the matrix in (11) is finite if and only if two elements of the matrix in (11) is analytic on the finite part of the $s = j\omega - axis$ (this analyticity will be called J-analyticity in the sequel) and $< O(s^{-1})$. When $vecd\,\Delta_{opt}$ is used, the first element becomes $\{U_*\, vec\,T_{11}\}_-$, which is J-analytic and $< O(s^{-1})$. The second element is obviously J-analytic since $U_\perp$ and T_{11} are J-analytic. Next, it will be shown that $U_{\perp *}\, vec\,T_{11}$ is strictly proper. Since $[\,U \,\vdots\, U_\perp\,]$ is square inner, it follows that $U_{\perp *}\,U = 0$ and hence $U_{\perp *}(T_b^T\odot T_a) = 0$. Notice that $vec\,T_{11} = vec\,\tilde{P}_{11} - vec(P_{12}P_{22}^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21}) = vec\,\tilde{P}_{11} - (T_b^T\odot T_a)\,vecd\,(\Delta_\alpha\Delta_\psi^{-1})$. Hence $U_{\perp *}\, vec\,T_{11} = U_{\perp *}\, vec\tilde{P}_{11}$ and this is strictly proper by Assumption 4.

As for the finite-cost yielding formula, once the optimal stable Δ_{opt} is determined we can express the free parameter Δ in (5) as $\Delta = \Delta_{opt} + \Delta_f$ where Δ_f now becomes a free stable parameter. Substituting this Δ into the first element of the matrix in (11), we obtain $U_* vec\,T_{11} - \Omega\, vecd(\Delta_{opt} + \Delta_f) = \{U_* vec\,T_{11}\}_- - \delta_f$ where $\delta_f = \Omega\, vecd\Delta_f$. In order for this term to have finite L_2 norm, δ_f should be stable and $\leq O(s^{-1})$. Hence $vecd\Delta_f = \Omega^{-1}\delta_f$ and this yields the formula $vecd(\Delta(s)) = \Omega^{-1}(\{U_* vec\,T_{11}\}_+ + \{U_* vec T_{11}\}_\infty + \delta_f)$. The cost function equality $E = \tilde{E} + \|\delta_f\|^2$ comes from the fact that $\{U_* vec\,T_{11}\}_-$ and δ_f are orthogonal in L_2 space of complex valued functions.

Next, since $U_* vec\,T_{11} = b - c$, it follows that $vecd\Delta = \Omega^{-1}(\{b-c\}_+ + \{b-c\}_\infty + \delta_f) = \Omega^{-1}(\{b-c\}_+ - \{c\}_\infty + \delta_f) = \Omega^{-1}(\{b\}_+ - c + \{c\}_- + \delta_f) - vecd(\Delta_\alpha \Delta_\psi^{-1}) + \Omega^{-1}(\{b\}_+ + \{c\}_- + \delta_f)$. Hence, $vecd\,T(s) = vecd(\Delta_\theta \Delta_\alpha + \Delta_\theta \Delta_f \Delta_\psi) = \Delta_\theta \Delta_\alpha e_n + \Delta_\theta \Delta_\psi vecd\,\Delta = \Delta_\theta \Delta_\psi \Omega^{-1}(\{b\}_+ + \{c\}_- + \delta_f)$.

It will be shown next that $T(s)$ in (10) is strictly proper. The spectral factor $\Omega(s)$ will be obtained in a different form from the one in (8). Define $\hat{\Delta}_\theta = diag\{\hat{\theta}_i\}$, where $\hat{\theta}_i$ is an arbitrary strict Hurwitz polynomial whose degree is the same as that of θ_i. Then $\Delta_\theta = \hat{\Delta}_\theta \Delta_{\theta p}$ with $\Delta_{\theta p} = diag\{\hat{\theta}_i/\theta_i\}$. Notice that $\Delta_{\theta p}$ is bi-proper and $\hat{\Delta}_\theta^{-1}$ is stable. Define $\hat{\Delta}_\psi$ and $\Delta_{\psi p}$ similarly so that $\Delta_\psi = \hat{\Delta}_\psi \Delta_{\psi p}$ with $\Delta_{\psi p}$ bi-proper and $\hat{\Delta}_\psi^{-1}$ stable. Eq. (8) becomes $\Omega_* \Omega = (\Delta_{\psi *} \tilde{P}_{21*}^T \tilde{P}_{21}^T \Delta_\psi) \circ (\Delta_{\theta *} P_{22*}^{-1} P_{12*} P_{12} P_{22}^{-1} \Delta_\theta) = \Delta_{\psi *} \Delta_{\theta *} (\tilde{P}_{21*}^T \tilde{P}_{21}^T) \circ (P_{22*}^{-1} P_{12*} P_{12} P_{22}^{-1}) \Delta_\psi \Delta_\theta = \hat{\Delta}_{\psi *} \hat{\Delta}_{\theta *} \Phi_o \hat{\Delta}_\theta \hat{\Delta}_\psi$ wth $\Phi_o = (\Delta_{\psi p *} \tilde{P}_{21*}^T \tilde{P}_{21}^T \Delta_{\psi p}) \circ (\Delta_{\theta p *} P_{22*}^{-1} P_{12*} P_{12} P_{22}^{-1} \Delta_{\theta p})$. Consider the spectral factorization $\Phi_o = \Omega_{o*} \Omega_o$. Then it follows that $\Omega = \Omega_o \Delta_\theta \Delta_\psi$ and hence $vecd\,T = \Delta_\theta \hat{\Delta}_\theta^{-1} \Delta_\psi \hat{\Delta}_\psi^{-1} \Omega_o^{-1}(\{b\}_+ + \{c\}_- + \delta_f)$. In view of Lemma 4, it is not difficult to show that $\Omega_o^{-1} \leq O(s^{k_1+k_2+k_3})$, which is proper by Assumption 7. Since $\Delta_\theta \hat{\Delta}_\theta^{-1}$ and $\Delta_\psi \hat{\Delta}_\psi^{-1}$ are proper and $(\{b\}_+ + \{c\}_- + \delta_f)$ is strictly proper, $T(s)$ is strictly proper. Q.E.D.

4. Conclusion

In this paper, H_2 designs for the one-degree-of-freedom decoupling control systems are treated for the generalized plant. A structural assumption on the general plant is set to exploit the characteristics of the one-degree-of-freedom controller configuration and the class of all realizable closed loop transfer matrices is parameterized. The optimal H_2 controller is obtained together with the ones that yield finite H_2 cost functions under compact assumptions. It is shown that the optimal closed transfer matrix is strictly proper under the reasonable order assumptions on the generalized plant.

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Appendix

As a preliminary for proofs of Lemmas 1 and 3, we present the formula of all stabilizing controllers. When Assumption 1 is satisfied, the class of all stabilizing controllers can be parametrized. Let $P_{22} = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s)$ be polynomial coprime fractional expressions. There always exist polynomial matrices $X(s)$, $Y(s)$, $X_1(s)$ and $Y_1(s)$ such that $\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ with $\det X(s) \cdot \det X_1(s) \neq 0$. In this case the class of all stabilizing controllers is characterized by the YJB formula(Youla et. al., 1976) $C(s) = -(Y + A_1K)(X - B_1K)^{-1}$ with $K(s)$ arbitrary real rational stable matrices and $\det(X - B_1K) \neq 0$.

Proof of Lemma 1: In view of the above YJB formula, it can be concluded that a rational matrix T is realizable if and only if it is of the form $T = P_{22}(Y + A_1K)A$ which is obtained by inserting the YJB formula for $C(s)$ into the equation $T = -P_{22}C(I - P_{22}C)^{-1}$. (Necessity) When T is of the form $T = P_{22}(Y + A_1K)A$, K being stable, it is obvious that $P_{22}^{-1}T$ and $T = B_1A_1^{-1}(YA + A_1KA)$ are stable, where use is made of the fact that $YA = A_1Y_1$. Since $P_{22}^{-1}TA^{-1} = Y + A_1K$ is stable, $P_{22}^{-1}TP_{22}$ is stable. Since $I - T = (X - B_1K)A$, $(I - T)A^{-1}$ and hence $(I - T)P_{22}$ are stable and $\det(I - T) \neq 0$ because of the constraint $\det(X - B_1K) \neq 0$. (Sufficiency) For a given rational matrix $T(s)$, suppose that $T, P_{22}^{-1}T, P_{22}^{-1}TP_{22}$ and $(I - T)P_{22}$ are stable and $\det(I - T) \neq 0$. Resolving the equation $T = P_{22}(Y + A_1K)A$ for K, we obtain $K = A_1^{-1}P_{22}^{-1}TA^{-1} - A_1^{-1}Y$. If this K is stable, then T is realizable. First, $A_1K = P_{22}^{-1}TA^{-1} - Y$ is stable since $P_{22}^{-1}TP_{22}$ is stable. Next, $B_1K = TA^{-1} - P_{22}Y = TA^{-1} - A^{-1} + X = X - (I - T)A^{-1}$ is stable since $(I - T)P_{22}$ is stable. That A_1K and B_1K are stable implies that K is stable. The constraint $\det(I - T) \neq 0$ assures that $\det(X - B_1K) \neq 0$.

Proof of Lemma 3: First notice that Assumption 3 is equivalent to the condition that $\tilde{P}_{11} - P_{12}A_1Y_1\tilde{P}_{21}$, $P_{12}A_1$ and $A\tilde{P}_{21}$ are stable (Nett, 1986). From the definition of Δ_θ, $P_{22}^{-1}\Delta_\theta$ and hence $B_1^{-1}\Delta_\theta$ are stable so that $T_a = P_{12}P_{22}^{-1}\Delta_\theta = P_{12}A_1B_1^{-1}\Delta_\theta$ is stable. From the definition of Δ_ψ, $\Delta_\psi P_{22}$ and hence $\Delta_\psi A^{-1}$ are stable so that $T_b = \Delta_\psi\tilde{P}_{21} = \Delta_\psi A^{-1}A\tilde{P}_{21}$ is stable. Next, $T_{11} = \tilde{P}_{11} - P_{12}P_{22}^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21} = \tilde{P}_{11} - P_{12}A_1B_1^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21}$. Inserting the equalities $X_1A_1B_1^{-1} + Y_1 = B_1^{-1}$ and $\Delta_\theta\Delta_\alpha = I - \Delta_\beta\Delta\psi$ yields $T_{11} = \tilde{P}_{11} - P_{12}A_1Y_1\Delta_\theta\Delta_\alpha\tilde{P}_{21} - P_{12}A_1X_1P_{22}^{-1}\Delta_\theta\Delta_\alpha\tilde{P}_{21} = \tilde{P}_{11} - P_{12}A_1Y_1\tilde{P}_{21} + P_{12}A_1Y_1\Delta_\beta\Delta_\psi\tilde{P}_{21} - P_{12}A_1X_1P_{22}^{-1} \cdot \Delta_\theta\Delta_\alpha A^{-1}A\tilde{P}_{21}$, which is stable since $\tilde{P}_{11} - P_{12}A_1Y_1\tilde{P}_{21}$ is stable by Assumption 3 and $P_{22}^{-1}\Delta_\theta\Delta_\alpha P_{22}$ and hence $P_{22}^{-1}\Delta_\theta\Delta_\alpha A^{-1}$ are stable when P_{22} admits a decoupling controller (Lemma 2).