Novel 2D FDTD Scheme with Isotropic Dispersion Characteristics

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Abstract: A two dimensional (2-D) finite-difference time-domain (FDTD) method based on a novel finite difference scheme is developed to eliminate the numerical dispersion errors. In this paper, numerical dispersion and stability analysis of the new scheme are given, which show that the proposed method is nearly dispersionless, and stable for a larger time step than the standard FDTD method.

Keywords: FDTD, Isotropic Dispersion, Stability

1. INTRODUCTION

For past decades finite-difference time-domain (FDTD) method has been very popular and widely used for a wide range of applications since it provides many advantages such as low computational complexity, great flexibility, easy implementation, etc. However, the standard FDTD algorithm has suffered from the so-called numerical dispersion which makes wave propagate at different velocity dependant on the propagation direction. Since very small cells should be used to reduce the dispersion error which will be accumulated with increasing time, the FDTD scheme may generally be used for electrically small size problems. To rectify the numerical anisotropic dispersion, several techniques have been proposed which are based on higher-order differential scheme, non-standard differential operator, introduction of an artificial anisotropy, overlapped lattices, etc. [1]. However, these methods may increase overall algorithm complexity at expense of reducing the dispersion.

In this paper, a novel 2D FDTD algorithm is proposed whose dispersion error can be controlled. The proposed scheme is based on an observation that the anisotropic dispersion of the standard FDTD may be caused by not sufficient spatial samplings to approximate the spatial derivatives. Hence, in the paper, fields are sampled at more points, 8 different points (4 standard Yee points + 4 additional points) in an isotropic manner, which results in a weighted sum of two different numerical derivative schemes to approximate the spatial derivatives (see Figure 1). To use the standard Yee grid, the fields at 4 additional points are estimated by a linear interpolation of fields in two adjacent cells. By varying the weighting factor, the overall dispersion of the proposed algorithm can be controlled; especially, a nearly isotropic dispersion can be obtained. Further by removing the numerical ether, the exact phase velocity can be achieved. Additionally the proposed method can relax the Courant Friedrich Levy (CFL) stability constraint of the standard FDTD, which is stable for a larger time step.

2. FORMULATION

The standard Yee algorithm approximates the spatial derivative using the second order central finite difference, (1). Another spatial derivative scheme is also known of same order of accuracy, which is given by (2). To keep using the Yee's grid, the values at each 4 points are calculated by linearly interpolating two values in adjacent points as shown in figure 1, which yields (3).

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\mathbf{d}}_x^2 f_{i,j}}{\Delta x} = \frac{f_{i+1/2,j} - f_{i-1/2,j}}{\Delta x} \quad (1)$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\mathbf{d}}_x^4 f_{i,j}}{2\Delta x} = \frac{\tilde{\mathbf{d}}_x^2 f_{i,j+1/2} + \tilde{\mathbf{d}}_x^2 f_{i,j-1/2}}{2\Delta x} \quad (2)$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\mathbf{d}}_x^6 f_{i,j}}{\Delta x} = \frac{\tilde{\mathbf{d}}_x^2 f_{i,j+1} + 2\tilde{\mathbf{d}}_x^2 f_{i,j} + \tilde{\mathbf{d}}_x^2 f_{i,j-1}}{4\Delta x} \quad (3)$$

Note that the superscript of the difference operators indicates the number of sampling points used for approximating the derivative.

Substituting (3) into the standard Yee algorithm, a new scheme can be formulated whose numerical dispersion is more anisotropic than the Yee method. However, the behavior of the anisotropy is opposed to that of the standard Yee method. Hence, by combining the new and standard methods the anisotropy of the dispersion can be controlled: in the new scheme the spatial derivatives are replaced by a weighed sum of $\tilde{\mathbf{d}}_u^2$ and $\tilde{\mathbf{d}}_u^6$. As an example, Maxwell's equation for the transverse magnetic (TM) wave can be discretized as seen in (4) where α is a weighting factor which has to be determined.

$$\tilde{\mathbf{d}}_t H_x^n(i,j+1/2) = -\alpha \frac{\Delta t}{4\mu\Delta y}\tilde{\mathbf{d}}_y^6 E_z^n(i,j+1/2) - (1-\alpha)\frac{\Delta t}{\mu\Delta y}\tilde{\mathbf{d}}_y E_z^n(i,j+1/2)$$

$$\tilde{\mathbf{d}}_t H_y^n(i+1/2,j) = \alpha \frac{\Delta t}{4\mu\Delta x}\tilde{\mathbf{d}}_x^6 E_z^n(i+1/2,j) + (1-\alpha)\frac{\Delta t}{\mu\Delta y}\tilde{\mathbf{d}}_y E_z^n(i+1/2,j)$$

$$\tilde{\mathbf{d}}_t E_z^{n+1}(i,j) = \begin{bmatrix} \alpha \frac{\Delta t}{4\varepsilon\Delta x}\tilde{\mathbf{d}}_x^6 H_y^{n+1/2}(i,j) + (1-\alpha)\frac{\Delta t}{\varepsilon\Delta x}\tilde{\mathbf{d}}_x H_y^{n+1/2}(i,j) \\ -\alpha \frac{\Delta t}{4\varepsilon\Delta y}\tilde{\mathbf{d}}_y^6 H_x^{n+1/2}(i,j) - (1-\alpha)\frac{\Delta t}{\varepsilon\Delta y}\tilde{\mathbf{d}}_y H_x^{n+1/2}(i,j) \end{bmatrix} \quad (4)$$

where $\tilde{\mathbf{d}}_t$ is the central time difference scheme.

3. DISPERSION AND STABILITY ANALYSIS

Following the procedure in [2], the numerical dispersion relation of the proposed scheme can be derived assuming $\Delta x = \Delta y = \Delta$.

$$f = f_{Yee} + \frac{1}{\Delta^2} C_+ C_\times \left(\alpha - \frac{2}{C_+} \right)^2 - \frac{1}{\Delta^2} \frac{4C_\times}{C_+} = 0 \quad (5)$$

where f_{Yee} denotes the dispersion relation of the standard FDTD [2]. C_+ and $C_\times$ are defined as

$$C_+ = \sin^2\left(\frac{\tilde{k}_x\Delta}{2}\right) + \sin^2\left(\frac{\tilde{k}_y\Delta}{2}\right),\quad C_\times = \sin^2\left(\frac{\tilde{k}_x\Delta}{2}\right)\sin^2\left(\frac{\tilde{k}_y\Delta}{2}\right)$$

As seen in (5), the dispersion error is a function of α as well as Δ, and Δ_t. It is very difficult to determine α to minimize the fluctuation of the dispersion over all propagation directions. Hence, first it is investigated the variation of α as a function of azimuth angles. Since (5) is a quadratic equation about α, the exact solution of (5) can be obtained analytically. Figure 2 shows the variation of α as a function of the cell size. As seen in the figure, for mostly adopted cell size, $\lambda/10 \sim \lambda/20$, α is not varied much and hence can be considered constant, and optimal value. This indicates the optimal α makes the dispersion almost zero (dispersionless) in all directions independently on the other parameters such as Δ and Δ_t. Thus the proposed scheme provides almost isotropic dispersion with the optimal α. Figure 3 shows the computed dispersion with the estimated optimal α, which is almost isotropic as expected. It can be observed that the phase velocity of the proposed scheme is slower than that of the Yee scheme, but by artificially reducing the cell size (forcing to equalize the numerical phase velocity to the ideal velocity) the numerical propagation constant can be made very close to the exact value over all azimuth angles. Next the effect of cell size (frequency) is considered on α for a wide band simulation. Figure 4 shows α as a function of the cell size, which indicates α is not varying much for a wide range of the cell size. Therefore the proposed scheme is also suitable to a wide band simulation. Figures 2 and 4 also show α is very insensitive to $S = c\Delta_t / \Delta$.

Next the stability of the proposed scheme is analyzed. Assuming that α is constant and substituting α calculated at $\varphi = 0^o$ into (5), (5) can be expressed simply as

$$f = f_{Yee} - E_{Yee} = -\frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) + \frac{1}{\Delta^2}\sin^2\left(\frac{2}{\Delta}\sin^{-1}\left(\frac{\Delta}{c\Delta t}\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right)\right)\right) \quad (6)$$

where $\tilde{\omega}$ denotes a numerical complex frequency. In (6), $\tilde{\omega}$ must be real to guarantee that the field component does not exponentially grow at every time step. Thus the stability criterion is simply given by $c\Delta t \le \Delta$. Therefore the proposed algorithm may be stable for larger time steps than for the standard Yee scheme. However, it should be pointed out that this criterion is not obtained rigorously, so that in practical situations the stability condition could be less than the estimated value.

4. NUMERICAL RESULTS

To demonstrate the validity of the proposed FDTD method, a simple problem is considered. An air-filled cavity enclosed by a perfect conductor whose size is 400×400 cells is selected for both the proposed algorithm and the standard Yee algorithm. A Gaussian pulse with a center frequency $f_o = 3GHz$ and a bandwidth $BW = 1.5GHz$ is excited at the center of the cavity. Grid size is chosen to be $\lambda_{\min}/20$, where $\lambda_{\min}$ is the wavelength of the highest frequency. Figure 5 shows the E_z field component calculated at the source point. It is shown in the figure that the proposed scheme is stable beyond the CFL limit, $S = 1/\sqrt{2} = 0.707$ for a square Yee grid in 2D FDTD. In figure 6, the results of the proposed scheme are compared, which uses the stretched coordinate to remove the artificial dielectric effect. As expected, the Figure shows that the time-domain pulse of the stretched case is slightly faster than those of the Yee and the unstretched case.

5. CONCLUSIONS

A novel FDTD method to reduce the numerical dispersion error is proposed by combining two spatial difference operators that sample fields at 6 points in the space. Introducing the stretched coordinate, an almost dispersionless FDTD scheme is developed, and it is also shown that the new method is stable beyond the CFL limit. For a cavity problem, the proposed algorithm is verified comparing with the standard Yee algorithm.

6. REFERENCE

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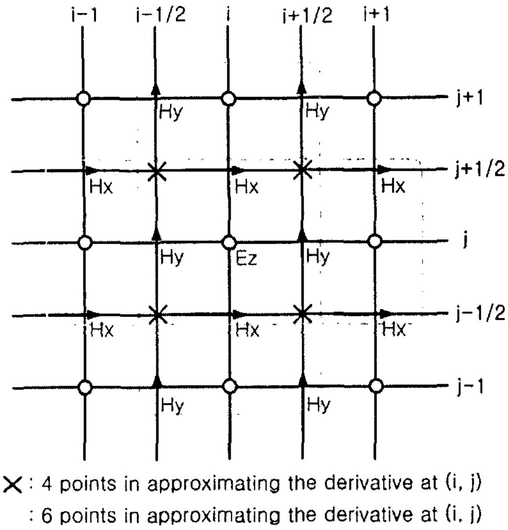

Figure1. Yee grid for 2D problem algorithm.

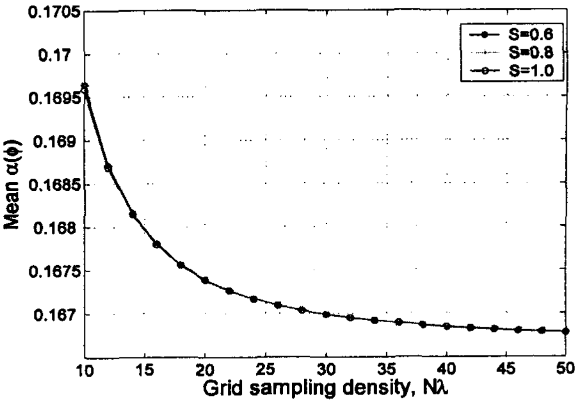

Figure 4. The mean of optimum weighting factor versus the grid sampling density for s=0.6, 0.8, 1.0.

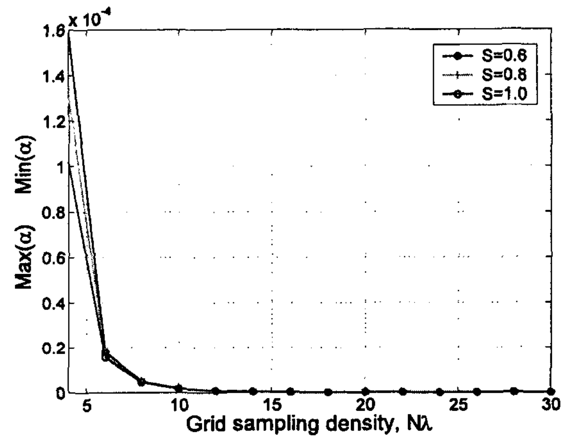

Figure 2. The maximum variation of weighting factor versus the grid sampling density for s=0.6, 0.8, 1.0.

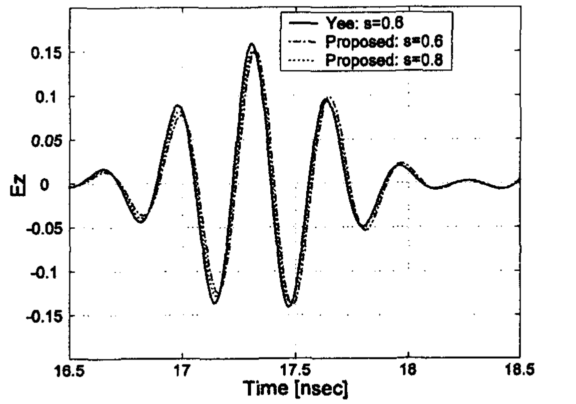

Figure 5. Comparison of E_z field inside the cavity for the standard Yee (s=0.6) and the proposed scheme (s=0.6, s=0.8)

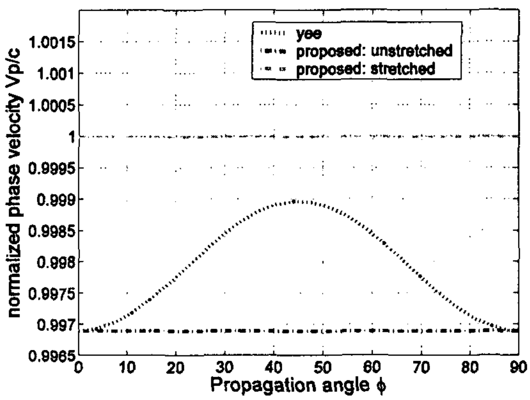

Figure 3. Comparison of normalized phase velocity of the standard Yee algorithm and the proposed algorithm with s=0.5, α=0.1674, Δ=λ/20.

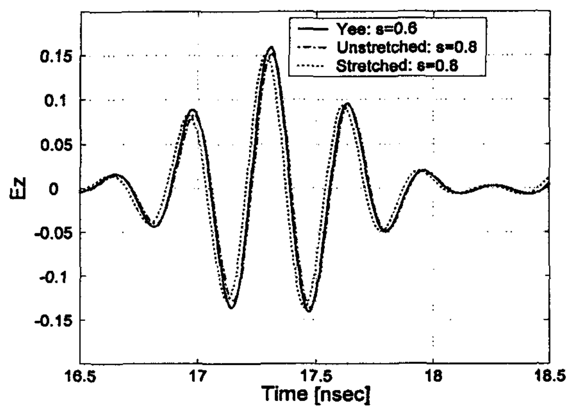

Figure 6. Copamrison of E_z field inside the cavity for standard Yee (s=0.6) and proposed scheme of stretched and unstretched case (s=0.8)

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