Some relation between compact set-valued functionals and compact set-valued Choquet integrals

Lee Chae Jang, * Hyun Mee Kim
Dept. of Mathematics and Computer Science, Konkuk University
* Dept. of Mathematics, Kyunghee University

Abstract

In this paper, we consider comonotonically additive compact set-valued functionals instead of interval-valued functionals and study some characterizations of them. And we also investigate some relation between compact set-valued functionals and compact set-valued Choquet integrals.

Key words : fuzzy measures, Choquet integrals, compact set-valued functionals, comonotonically additive

1. Introduction

In this paper, we consider Choquet integrals of compact set - valued functions. We note that Jang [4] studied closed set-valued Choquet integrals with respect to fuzzy measures. In Section 2, we list various definitions and notations which are used in the proof of our results. In Section 3, we will define comonotonically additive compact-valued functionals, and prove some properties of them. And we also investigate some relation between compact set-valued functionals and compact set-valued Choquet integrals.

2. Preliminaries and definitions

Throughout this paper, we assume that X is a locally compact Hausdorff space, $K(K^+)$ is the class of continuous(non-negative) functions on X with compact support, Ω is the σ-algebra subsets of X. The class of measurable functions is denoted by M and the class of non-negative measurable functions is denoted by M^+.

A non-additive measure on a measurable space (X, Ω) is an extended real-valued function $\mu : \Omega \to [0, \infty]$ satisfying

(1) $\mu(\varnothing) = 0$,

(2) $\mu(A) \le \mu(B)$ wherever $A, B \in \Omega$, $A \subset B$.

A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets, we have

$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1) < \infty$, we have

$$\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

If μ is lower and upper semi-continuous, it is said to be continuous(see [6]).

We note that if $\mu(X) < \infty$, we define the conjugate μ^C of μ by $\mu^C(A) = \mu(X) - \mu(A^C)$, where A^C is the complement of $A \in \Omega$.

Recall that a function $f : X \to [0, \infty]$ is said to be measurable if $\{x \mid f(x) > \alpha\} \in \Omega$ for all $\alpha \in (-\infty, \infty)$ (see [1,2,3]).

Definition 2.1 [4,5,8] (1) The Choquet integral of measurable function $f \in M^+$ with respect to a fuzzy measure μ is defined by

$$(C)\int f d\mu = \int_0^{\infty} \mu\{x| f(x) > r\} dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet of f can be defined and its value is finite.

Throughout this paper, R^+ will denote the interval $[0, \infty)$, $C_0(R^+) = \{A | A$ is compact subset of $R^+\}$. Then an element in $C_0(R^+)$ is called an interval number. On the interval number set, we define; for each pair $A, B \in C_0(R^+)$ and for all $c \in R^+$,

$$A + B = \{a + b | a \in A, b \in B\},$$

$$cA = \{ca | a \in A\}, \text{ and}$$

$A \le B$ if and only if (1) for every $a \in A$, there exists $b \in B$ such that $a \le b$ and (2) for every $b \in B$, there exists $a \in A$ such that $a \le b$.

We note that $(C_0(R^+), d_H)$ is a metric space, where of the Hausdorff metric defined by

$d_H(A, B) = max\{sup_{a \in A} inf_{b \in B} | a - b|,$ $\{sup_{b \in B} inf_{a \in A} | a - b| \}$ for all $A, B \in C_0(R^+)$.

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a compact set-valued function $F: X \to C(R^+) \setminus \{\emptyset\}$.

Definition 2.3 [4,5] Let f and g be measurable non-negative functions. We say that f and g are comonotonic, in symbol $f \sim g$, if and only if $f(x) < f(x') \to g(x) \le g(x')$ for all $x, x' \in X$.

Theorem 2.4 [6] Let f, g and h be measurable functions. Then we have

(1) $f \sim f$,

(2) $f \sim g \to g \sim f$,

(3) $f \sim k$ for all $k \in R^+$,

(4) $f \sim g$ and $f \sim h \to f \sim (g + h)$.

Definition 2.5 A compact set-valued function F is said to be measurable if for each open set $O \subset R^+$, $F^{-1}(O) = \{x | F(x) \cap O\} \neq \emptyset \in \Omega$.

Definition 2.6 Let F be a compact set-valued function. A measurable function $f: X \to R^+$ satisfying $f(x) \in F(x), \forall x \in X$ is called a measurable selection of F.

Definition 2.7 Let F be a compact set-valued function, let μ be a non-additive measure and $A \in \Omega$.

(1) The Choquet integral of F on A is defined by

$$(C)\int F d\mu = \{(C)\int_A f d\mu \,|\, f \in S(F)\},$$

where $S(F)$ is the family of $\mu - a.e.$ measurable selections of F, that is, $S(F) = \{f \in M^+ | f(x) \in F(x), x \in X, \mu - a.e.\}$.

(2) A compact set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \neq \emptyset$, it is said to be Choquet integrable if $(C)\int F d\mu$ exists and dose not include ∞.

(3) A compact set-valued function F is said to be Choquet integrably bounded if there is a function $g \in M^+$ such that

$$F(x) = sup_{r \in F(x)} r \le g(x), \forall x \in X.$$

We consider the following classes of closed set-valued functions and compact set-valued functions;

$\tau = \{F | F: X \to C(R^+) \setminus \{\emptyset\}$ is measurable closed set-valued function and Choquet integrably bounded$\}$,

$\tau_1 = \{F | F: X \to C(R^+) \setminus \{\emptyset\}$ is measurable compact set-valued function and Choquet integrably bounded $\}$.

We recall that $cl(A)$ means the closure of subset A of R^+.

We consider the following non-negative functional on K^+ and introduce theorems which are used to discuss some characterizations of compact-valued functionals.

Definition 2.8 [6] Let l be a real-valued functional on K^+.

(1) l is comonotonically additive if and only if $f \sim g \rightarrow l(f+g) = l(f) + l(g), \quad \forall f, g \in K^+$.

(2) l is positively homogeneous if and only if $l(kf) = kl(f),\ \forall f \in K^+,\ k \in R^+$.

(3) l is monotonic if and only if $f \leq g \rightarrow \quad l(f) \leq l(g),\ \forall f, g \in K^+$.

Definition 2.9 [6] A fuzzy measure μ is said to be outer regular if and only if for all $B \in \Omega$, $\mu(B) = inf\{\mu(O) \mid O$ is an open set such that $O \supset B\}$.

We remark that if μ is a continuous fuzzy measure on (X, Ω), then it is outer regular.

Theorem 2.10 [6] For comonotonically additive, positively homogeneous, and monotonic functional l on K^+, there exists a outer regular fuzzy measure μ on Ω such that for all $f \in K^+$,

$$l(f) = (C)\int f d\mu.$$

Definition 2.11 Let $G, H \in \mathbb{T}$. We define

$$(G \cup H)(x) = G(x) \cup H(x) \quad \forall x \in X.$$

Theorem 2.12 (1) If $G, H \in \mathbb{T}$, then we have $G \cup H \in \mathbb{T}$ and $G \cap H \in \mathbb{T}$.

(2) If $G, H \in \mathbb{T}_1$, then we have $G \cup H \in \mathbb{T}_1$ and $G \cap H \in \mathbb{T}_1$.

Theorem 2.13 [7] If $F \in \mathbb{T}_1$, then we have

$$(C)\int F d\mu \text{ is compact.}$$

Theorem 2.14 [7] If $F, G \in \mathbb{T}_1$, then we have

$$(C)\int (F \cup G) d\mu = (C)\int F d\mu \cup (C)\int G d\mu.$$

Theorem 2.15 [7] If $F, G \in \mathbb{T}_1$, then we have

$$(C)\int (F \cap G) d\mu \subset (C)\int F d\mu \cap (C)\int G d\mu.$$

Definition 2.16 [7] Let $G, H \in \mathbb{T}$. Then an addition operation $\uplus$ on $\mathbb{T}$ is defined by

$$G \uplus H = cl(G \cup H).$$

We easily obtain the following property of this operation $\uplus$ on $\mathbb{T}$.

Theorem 2.17 [7] If $F, G \in \mathbb{T}_1$, then

$$(C)\int (F \uplus G) d\mu = (C)\int F d\mu \uplus (C)\int G d\mu.$$

Using these operations, we can define another addition operation on $\mathbb{T}$.

Definition 2.18 Let $G, H \in \mathbb{T}$. Then an addition operation $\oplus$ on $\mathbb{T}$ is defined by

$$G \oplus H = cl(G + H).$$

Definition 2.19 [6] Let $F, G \in \mathbb{T}$. We say that F and G are comonotonic, in symbol, $F \sim G$ if and only if

(1) $f^*(x) < f^*(x') \rightarrow g^*(x) \leq g^*(x')$ for all $x, x' \in X$, and

(2) $f_*(x) < f_*(x') \rightarrow g_*(x) \leq g_*(x')$ for all $x, x' \in X$, where $f^*(x) = sup\{r \mid r \in F(x)\}$, $f_*(x) = inf\{r \mid r \in F(x)\}$, $g^*(x) = sup\{r \mid r \in G(x)\}$, and $g_*(x) = inf\{r \mid r \in G(x)\}$.

From Definition 2.19, clearly we have the following theorem.

Theorem 2.20 Let F, G and $H \in \mathbb{T}_1$. Then we have

(1) $F \sim F$,

(2) $F \sim G \rightarrow G \sim F$,

(3) $F \sim A$ for all $A \in C(R^+)$,

(4) $F \sim G$ and $F \sim H \rightarrow F \sim G \oplus H$.

3. Main Results

In this section, we consider the following class of compact set-valued functions with continuous selections;

$$\mathbb{T}_2 = \{F \in \mathbb{T}_1 \mid S(F) \subset K^+\}.$$

Definition 3.1 (1) A mapping $T: \mathbb{T}_2 \to C(R^+)$ is said to be a compact-valued functional.

(2) T is comonotonically additive if and only if $F \sim G \to T(F \oplus G) = T(F) \oplus T(G)$ $\forall F, G \in \mathbb{T}_2$.

(2) T is positively homogeneous if and only if $T(kF) = kT(F)$, $k \in R^+$, $\forall F \in \mathbb{T}_2$.

(3) T is monotonic if and only if $F \leq G \to T(F) \leq T(G), \forall F, G \in \mathbb{T}_2$

Definition 3.2 Let $l: K^+ \to R^+$ be a real-valued functional. A mapping $T_l: T_2 \to C(R^+)$ is said to be a compact-valued functional induced by l if for all $F \in \mathbb{T}_2$,

$$T_l(F) = \{l(f) \mid f \in S(F)\}.$$

Theorem 3.3 Let be $l: K^+ \to R^+$ a continuous real-valued functional and a compact-valued functional $T_l: \mathbb{T}_2 \to C(R^+)$ induced by l. If l is comonotonically additive and $F \sim G$, then we have

(1) $T_l(F \oplus G)$ is a compact,

(2) $T_l(F \oplus G) = T_l(F) \oplus T_l(G)$ $\forall F, G \in \mathbb{T}_2$

Theorem 3.4 Let be $l: K^+ \to R^+$ a continuous real-valued functional and a compact-valued functional $T_l: \mathbb{T}_2 \to C(R^+)$ induced by l. Then we have

(1) $T_l(F)$ is a compact for all $F \in \mathbb{T}_2$,

(2) If l is monotonic $\to T_l$ is monotonic,

(3) If l is positively homogeneous $\to T_l$ is positively homogeneous,

(4) $T_l(F) \uplus T_l(G) = T_l(F \uplus G)$ for all $F, G \in \mathbb{T}_2$.

Theorem 3.3 (2) implies the following corollary.

Corollary 3.5 Let be $l: K^+ \to R^+$ a continous real-valued functional and a compact-valued functional $T_l: \mathbb{T}_2 \to C(R^+)$ induced by l. If l is comonotonically additive , then T_l is comonotonically additive.

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