Proceedings of KFIS Autumn Conference 2005 Volume 15, Number 2

쇼케이적분을 이용한 구간치 퍼지수 상의 거리측도에 관한 성질

Some algebraic properties and a distance measure for interval-valued fuzzy numbers

*장이채 · **김현미

*Lee Chae Jang · **Hyun Mee Kim

*Dept. of Mathematics and Computer Science, Konkuk University

**Dept. of Mathematics, Kyunhee University

요약

퍼지측도와 관련된 폐집합치 쇼케이적분에 대해 장에 의해 연구되어 왔음을 알 수 있다. 본 논문에서는 컴팩트 집합치 함수의 쇼케이적분을 생각하고 이와 관련된 성질들을 조사한다. 특히, 구간치 함수 대신에 컴팩트 집합치 함수를 이용하여 컴팩트 집합치 쇼케이적분의 특성들을 조사한다.

Abstract

Interval-valued fuzzy sets were suggested for the first time by Gorzalczang(1983) and Turken(1986). Based on this, Wang and Li extended their operations on interval-valued fuzzy numbers. Recently, Hong(2002) generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. In this paper, we define a distance measure on interval-valued fuzzy numbers using Choquet integral with respect to a classical measure and investigate their properties.

Key words : Interval-valued fuzzy number, Distance measure, Choquet integral.

1. Introduction

Interval-valued fuzzy sets were suggested for the first time by Gorzalczany[4] and Turksen[8]. Based on this Wang and Li definded fuzzy numbers and gave their extended operations. Recentely Hong generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. In this paper, we propose the concept of Choquet integral of fuzzy numbers with respect to metric instead of Riemann integral and we consider some properties of interval-valued fuzzy numbers.

In Section 2 we give preliminary definitions which are required in the following discussion.. Section 3 we deal some properties of fuzzy numbers. Section 4 using these define Choquet distance of fuzzy numbers and investigate some properties.

2. Definitions and Preliminaries

Throughout this paper, R^+ will denote the interval $[0,\infty)$,

$$[I]=\{[a,b] \mid a,b \in R^+ \text{ and } a \le b\}.$$

Then an element in $[I]$ is called an interval number.

Definition 2.1. If $a_t \in I, t \in T$, then we define

$\vee_{t\in T}a_t = sup\{a_t : t\in T\}$,

$\wedge_{t\in T}a_t = inf\{a_t : t\in T\}$.

We also define for $[a_t, b_t]\in [I]$, $t\in T$,

$\vee_{t\in T}[a_t, b_t] = [\vee_{t\in T}a_t, \vee_{t\in T}b_t]$,

$\wedge_{t\in T}[a_t, b_t] = [\wedge_{t\in T}a_t, \wedge_{t\in T}b_t]$.

Definition 2.2. Let $[a_1, b_1], [a_2, b_2]\in [I]$ and $k\in R^+$. We define

$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$

$[a_1, b_1]\cdot [a_2, b_2] = [a_1\cdot a_2, b_1\cdot b_2]$

$k[a_1, b_1] = [ka_1, kb_1]$,

$[a_1, b_1] \le [a_2, b_2]$ if and only if $a_1 \le a_2$ and $b_1 \le b_2$,

$[a_1, b_1] < [a_2, b_2]$ if and only if $[a_1, b_1] \le [a_2, b_2]$ but $[a_1, b_1] \ne [a_2, b_2]$.

Definition 2.3. Let X be an ordinary nonempty set, then the mapping $A : X\to [I]$ is called an interval-valued fuzzy set on X. All interval-valued fuzzy set on X are denoted by $IF(X)$.

Definition 2.4. Let $A\in IF(X)$, let $A(X) = [A_-(X), A^-(X)]$, where $x\in X$. Then two ordinary fuzzy set $A_- : X\to [I]$ and $A^- : X\to [I]$ are called lower fuzzy set and upper fuzzy set of A, respectively, simply write $A = [A^-, A^+]$.

Definition 2.5. Let $A\in IF(X)$, $[\lambda_1, \lambda_2]\in [I]$. Then we call

$A_{[\lambda_1,\lambda_2]} = \{x\in X \mid A_-(X) \ge \lambda_1,\ A^-(X)\ge \lambda_2\}$

and

$A_{(\lambda_1,\lambda_2)} = \{x\in X \mid A_-(X) > \lambda_1, A^-(X) > \lambda_2\}$

the $[\lambda_1, \lambda_2]$-level set of A and the (λ_1, λ_2)-level set of A, respectively. And let $A_{\lambda^-}(x) = \{x\in X \mid A_-(X) > \lambda\}$ and $A_{\lambda^+}(x) = \{x\in X \mid A^-(X) > \lambda\}$.

Definition 2.6. Let $A\in IF(X)$, $[\lambda_1, \lambda_2]\in [I]$. We define

$([\lambda_1, \lambda_2]A)(x) = [\lambda_1, \lambda_2]\wedge [A_-(X), A^-(X)]$.

Definition 2.7. Let $A\in IF(X)$, i.e., $A : R\to [I]$. Assume that following conditions are satisfied;

(1) A is normal, i.e., there exists $x_0\in R$ s.t. $A(x_0) = 1$.

(2) For arbitrary $[\lambda_1, \lambda_2]\in [I]-\{0\}$, $A_{[\lambda_1,\lambda_2]}$ is closed bounded interval.

Then we call A an interval-valued fuzzy number on real line R.

Let $IF^*(R)$ denoted the set of all interval-valued fuzzy numbers on real line R, and we write $[I]^+ = [I]-\{0\}$.

Definition 2.8. Let $A\in IF^*(X)$. Then A is called an interval convex fuzzy set, if for any $x, y\in R$ and $\lambda\in [0, 1]$, we have $A(\lambda x + (1-\lambda)y) \ge A(x)\wedge A(y)$.

Definition 2.9. Let $A, B\in IF^*(X)$. $\bullet\in\{+, -, \cdot, \div\}$. We define their extended operations to

$(A\bullet B)(z) = \vee_{z=x\bullet y}(A(x)\wedge B(y))$.

For each $[\lambda_1, \lambda_2]\in [I]^+$, we write $A_{[\lambda_1,\lambda_2]}\bullet B_{[\lambda_1,\lambda_2]} = \{x\bullet y : x\in A_{[\lambda_1,\lambda_2]}, y\in B_{[\lambda_1,\lambda_2]}\}$

Definition 2.10. Let $A\in IF^*(X)$. Then A is called a positive interval-valued fuzzy number, if $A(x) = \bar{0}$ whenever $x\le 0$; an A is called a negative interval-valued fuzzy number, if $A(x) = \bar{0}$ whenever $x\ge 0$.

All positive interval-valued fuzzy numbers and all negative interval-valued fuzzy numbers are denoted by $IF_+{}^*(R)$ and $IF_-{}^*(R)$, respectively.

Definition 2.11. (1) A fuzzy measure on a measurable space (X, A) is an extended real-valued function $\mu : A\to [0, \infty]$ satisfying

(i) $\mu(\phi) = 0, \mu(X) = 1$

(ii) whenever $A, B \in \mathcal{A}, A \subset B$, then $\mu(A) \leq \mu(B)$.
(2) μ is said to be continuous from below it for every increasing sequence $\{A_n\} \subset \mathcal{A}$ of measurable sets, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(3) μ is said to be continuous from above it for every decreasing sequence $\{A_n\} \subset \mathcal{A}$ of measurable sets, we have

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(4) if μ is said to be continuous from above and continuous from below, it is said to be continuous.

Definition 2.12. (1) The Choquet integral of a measure μ is defined by

$$(C)\int f d\mu = \int_0^{\infty} \mu_f(r) dr$$

where $\mu_f(r) = \mu\{x \mid f(x) > r\}$ and the integral on the right-hand side is an ordinary one.
(2) A measurable function f is called integrable if the choquet integral of f can be defined and its value is finite.

3. Interval-valued fuzzy numbers

Lemma 3.1. Let $A \in IF^*(X)$. Then A is normal if and only if A_- and A^- are normal fuzzy sets.

Lemma 3.2. Let $A \in IF^*(X)$. Then (2) is in definition 2.7 is equivalent to saying that for arbitrary $\alpha \in [0,1]-\{0\}$, $A_{-\alpha}$ and A^-_{α} are closed bounded intervals.

Lemma 3.2. Let $A \in IF^*(X)$. Then A is an interval convex fuzzy set if and only if A_- and A^- are convex fuzzy set.

It is noted that A_- (A^-) is convex if and only if for arbitrary $\alpha \in [0,1]$, $A_{-\alpha}$ (A^-_{α}) is convex set. Hence the following result which is a generalization of convex sets is convex.

Theorem 3.1. Let $A \in IF^*(X)$. Then A is an interval convex fuzzy set if and only if for any $[\lambda_1, \lambda_2] \in [I]^+$, $A_{[\lambda_1, \lambda_2]}$ is a convex set.

Theorem 3.2. Let $A, B \in IF^*(R)$, $\bullet \in \{+, -, \cdot, \div\}$. Then $(A \bullet B)(z) = [(A_- \bullet B_-)(z), (A^- \bullet B^-)(z)]$.

Corollary 3.3. [6] Let $A, B \in IF^*(R)$, Then $A+B$, $A-B$, $A \cdot B \in IF^*(R)$. Especially, $A \div B \in IF^*(R)$ whenever $B \in IF^*_-(R) \cup IF^*_+(R)$.

The following to important results immediate as an application of Theorem 3.2 and commutatively and associativity of fuzzy numbers under $+$ and $\bullet$ (see[2, 8])

Theorem 3.5. Let $A, B \in IF^*(R)$, Then $A \bullet B = B \bullet A$ where $\bullet \in \{+, \cdot\}$ and $A, B \in IF^*_-(R)$ or $A, B \in IF^*_+(R)$ whenever $\bullet$ choose.

Theorem 3.6. Let $A, B, C \in IF^*(R)$. Then $(A \bullet B) \bullet C = A \bullet (B \bullet C)$ where $\bullet \in \{+, \cdot\}$ and $A, B, C \in IF^*_-(R)$ or $A, B, C \in IF^*_+(R)$ whenever $\bullet$ choose.

4. Main Results

In this section at first we recall definition of fuzzy numbers on R and a well-known distance between fuzzy numbers.

Let $F^*(R)$ denote the set of all fuzzy set. $A: R \to [0,1]$ with the following properties:

(1) A_λ is compact (closed and bounded) for all $\lambda \in [0,1]$;

(2) A is normal, i.e., there exist $x_0 \in R$ such that $A(x_0)$; where $A_\lambda = \{x \in R \mid A(x) \geq \lambda\}$ is the λ-cut of A for $\lambda \in [0,1]$ and A_0 is the closure of the support set of A. Each $A \in F^*(R)$ is called a fuzzy number. The following is well-known metric in $F^*(R)$. [3]

Definition 4.1. For arbitrary fuzzy numbers $A, B \in F^*(R)$, the quantity

$$D_c(A,B) = (C)\int d_H(A_\lambda, B_\lambda)\, d\mu(\lambda)$$

$$= \int_0^\infty \mu\{\lambda \mid d_H(A_\lambda, B_\lambda) > \alpha\} d\alpha(\lambda)$$

is the Choquet distance between A and B, where d_H is the Hausdorff metric between A_λ and B_λ which is defined as

$$d_H(A_\lambda, B_\lambda)$$
$$= d_H(A_-(\lambda), B_-(\lambda)) \vee d_H(A^-(\lambda), B^-(\lambda))$$

since $A_-(\lambda)$ and $A^-(\lambda)$ the lower and the upper endpoint of A_λ, $A_\lambda = [A_-(\lambda), A^-(\lambda)]$.

Theorem 4.1. Let $A, B \in F^*(R)$. Then $D_C(A,B)$ is pseudo-metric.

Theorem 4.2. Let $A, B \in F^*(R)$ and A, B are continuous. Then $D_C(A,B) = 0$ if and only if $A = B$.

Theorem 4.3. Let $\{A_n\}$ is a sequence of fuzzy numbers and for each x, $A_n(x) = A(x)$. Then $\lim_{n\to\infty} D_C(A_n, A) = 0$.

Theorem 4.4. Let $A, B \in F^*(R)$. Then $D_C(\vee A_n, B) \leq \vee D_C(A_n, B)$.

4. References

[1] J. Aubin, "Set-valued analysis", Birkauser Boston, 1990.

[2] R .J. Aumann, "Integrals of set-valued functions", J. Math. Anal. ppl. Vol. 12, pp 1-12, 1965.

[3] Dug-Hun Hong and Sungho Lee, Some algebraic properties and a distance measure for interval-valued fuzzy numbers, Information Sciences Vol. 148, pp. 1-10, 2002.

[4] B. Gorzalczany, Approximate inference with interval-valued fuzzy sets - an outline, in Proceedings of Polish Symposium on Interval and Fuzzy Mathematics, Poznan, Poland, 1983, pp.89-95.

[5] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, "On comonotonically additive interval-valued Choquet integrals(II)", KFIS, Vol. 14, No. 1, pp.33-38, 2004.

[6] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, On set-valued Choquet integrals and convergence theorem (II), Bull. Kerean Math.Soc., Vol. 20, No. 1, pp.139-147, 2003.

[7] Lee-Chae Jang, Interval-valued Choquet integrals and their applications, Journal of Applied Mathematics and Computing, Vol. 16, No.1-2, pp.429-443, 2004.

[8] B. Turksen, Interval-valued fuzzy sets based on normal forms, Fuzzy Sets and Systems Vol. 20, pp. 191-210, 1986.

[9] G. Wang and X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems Vol. 98, pp. 331-335, 1998.