# A Strong LP Formulation for the Ring Loading Problem with Integer Demand Splitting

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**Abstract**

In this paper, we consider the Ring Loading Problem with integer demand splitting (RLP). The problem is given with a ring network, in which a required traffic requirement between each selected node pair must be routed on it. Each traffic requirement can be routed in both directions on the ring network while splitting each traffic requirement in two directions only by integer is allowed. The problem is to find an optimal routing of each traffic requirement which minimizes the capacity requirement. Here, the capacity requirement is defined as the maximum of traffic loads imposed on each link on the network. We formulate the problem as an integer program. By characterizing every extreme point solution to the LP relaxation of the formulation, we show that the optimal objective value of the LP relaxation is equal to $p$ or $p$+0.5, where $p$ is a nonnegative integer. We also show that the difference between the optimal objective value of RLP and that of the LP relaxation is at most 1. Therefore, we can verify that the optimal objective value of RLP is $p$+1 if that of the LP relaxation is $p$+0.5. On the other hand, we present a strengthened LP with size polynomially bounded by the input size, which provides enough information to determine if the optimal objective value of RLP is $p$ or $p$+1.

## 1. Introduction

In this paper, we consider the Ring Loading Problem with integer demand splitting (RLP). The problem is given with a ring network, in which a required traffic requirement between each selected node pair must be routed on it. Each traffic requirement can be routed in both directions on the ring network while splitting each traffic requirement in two directions only by integer is allowed. The problem is to find an optimal routing of each traffic requirement which minimizes the capacity requirement. Here, the capacity requirement is defined as the maximum of traffic loads imposed on each link on the network.

RLP was previously studied by Liese [3], Shyur et al.[5] and Lee and Chang [1]. They proposed heuristic algorithms and have not mentioned the computational complexity. Vachani et al.[7] recently presented a polynomial time algorithm for RLP. The complexity of their algorithm is $O(n^3)$, where $n$ is the number of nodes(links) on a cycle. Myung [4] gave an $O(n|K|)$-time algorithm, where $|K|$ is the number of selected node pairs. Very recently, Wang et al. [6] presented $O(|K|)$- time algorithm, if $|K| \ge n^{\varepsilon}$ for some small constant $\varepsilon > 0$.

Since RLP can be solved in polynomial time, it is natural to ask if there is a linear program whose number of variables and constraints is bounded by a polynomial function of the size of a given instance of RLP, and gives an optimal solution to RLP. In this paper, we try to give an answer to an interesting question which is closely related to the above question.

We first present an integer programming formulation of RLP in section 2. In section 3, we show that $z_{LP}$ is equal to $p$ or $p$+0.5, where $z_{LP}$ is the optimal objective value of the LP relaxation of the formulation and $p$ is a nonnegative integer. We also show that $z_{RLP} - z_{LP} \le 1$ by constructing a feasible solution to RLP, whose corresponding objective value is less than or equal to $\lfloor z_{LP} \rfloor + 1$, from an optimal extreme point solution to the LP relaxation which has fractional coordinates, where $z_{RLP}$ is the optimal objective value of RLP. Therefore, it is clear that $z_{RLP} = \lceil z_{LP} \rceil$ if $z_{LP}$ is equal to $p$+0.5, where $p$ is a nonnegative integer. On the other hand, if $z_{LP}$ is equal to $p$, then either $z_{RLP} = z_{LP}$ or $z_{RLP} = z_{LP} + 1$. Does there exist an LP which can tell if $z_{RLP} = z_{LP}$ or $z_{RLP} = z_{LP} + 1$. We give an affirmative answer to this question in section 4. Finally, we give concluding remarks in section 5.

## 2. Formulation of the Problem

The Ring Loading Problem with integer demand splitting (RLP) is defined on a ring $G=(V,E)$ with $V=\{1,2,\ldots,v\}$ and $E=\{(1,2),\ldots,(v-1,v),(v,1)\}$. The followings are the additional notations and definitions to be used in the formulation of RLP.

$K$ : set of selected node pairs (commodities),
$o_k, d_k$ : two nodes of a commodity *k*, for each $k\in K$, where $o_k<d_k$,
$r_k$ : traffic requirement of a commodity *k*, for each $k\in K$, assumed to be a positive integer,
$P_k^+$ : set of links which are used by the clockwise path of *k*, for each $k\in K$, i.e., $\{(o_k,o_k+1),(o_k+1,o_k+2),\ldots,(d_k-1,d_k)\}$,
$P_k^-$ : set of links which are used by the counter-clockwise path of *k*, for each $k\in K$, i.e., $E\setminus P_k^+$,
$x_k$ : the quantity of traffic requirement of commodity *k* which are routed in the clockwise direction, for each $k\in K$.

If we route $x_k$ units of traffic requirement of commodity $k$ in the clockwise direction, for each $k\in K$, $(r_k-x_k)$ units are routed in the counter-clockwise direction. Therefore, we can formulate RLP as follows :

$$\begin{aligned}\text{(RLP) min } & z\\ \text{s.t. } & x_k\le r_k,\ \forall k\in K,\\ & \sum_{\{k\in K|e\in P_k^+\}} x_k+\sum_{\{k\in K|e\in P_k^-\}}(r_k-x_k)\le z,\ \forall e\in E,\\ & x_k \text{ nonnegative integer},\ \forall k\in K.\end{aligned}$$

For ease of later expositions, let us define

$$I_e:\ -z+\sum_{\{k\in K|e\in P_k^+\}} x_k-\sum_{\{k\in K|e\in P_k^-\}} x_k\le R_e,$$

where $R_e=-\sum_{\{k\in K|e\in P_k^-\}} r_k$, and let $L_e(x,z)$ be the left-hand-side of $I_e$. Note that, $I_e$ is the inequality of the above formulation of RLP which corresponds to the link *e*, for all $e\in E$. For a feasible solution $(x^*,z^*)$, the value of the left-hand-side of $I_e$ is denoted by $L_e(x^*,z^*)$.

In the next section, we characterize extreme point solutions of the linear programming relaxation of (RLP) and analyze the strength of it.

### 3. Analysis of the LP Relaxation

Let (LP) be the linear programming relaxation of (RLP), that is, (RLP) without integrality restrictions. Let *P* be the set of feasible solutions to (LP) and $(\bar{x},\bar{z})$ be an extreme point of *P*. Let us define $EQ(\bar{x},\bar{z})$ be the set of defining inequalities of *P* which are satisfied at equalities by $(\bar{x},\bar{z})$ and $E(\bar{x},\bar{z})=\{e\in E\mid L_e(\bar{x},\bar{z})=R_e\}$. Further, define $\bar{K}(\bar{x})=\{k\in K\mid 0<\bar{x}_k<r_k\}$. Then by substituting the variables $\bar{x}_k$, $k\in K\setminus\bar{K}(\bar{x})$ into each inequality of $EQ(\bar{x},\bar{z})$, we can obtain the following system of linear equations :

$$-z+\sum_{\{k\in\bar{K}|e\in P_k^+\}} x_k-\sum_{\{k\in\bar{K}|e\in P_k^-\}} x_k=\bar{R}_e,\ \forall e\in E(\bar{x},\bar{z}) \quad (1)$$

where $\bar{R}_e$ is the updated right-hand side.

Let *B* be the left-hand side coefficient matrix of (1) and let *b* be the right-hand side vector of (1). After eliminating redundant equations, we can assume that *B* is an *m* by *m* nonsingular integral matrix and *b* is an *m* by 1 integral vector. Let $(\hat{B}:\hat{b})$ be the upper-triangular matrix obtained by applying Gaussian elimination procedure to $(B:b)$. We call an integer *c* is even (odd) if the absolute value of *c* is even (odd), from now on.

**Proposition 1.** *$(\hat{B}:\hat{b})$ has the following structure.*
*i) Each element of the first row of $\hat{B}$ is either 1 or -1, each nonzero element of the other rows of $\hat{B}$ is either 2 or -2, and $\hat{b}$ is an integral vector.*
*ii) If all elements of b are either even or odd, $\hat{b}_i$ is even, for each $2\le i\le m$.*
**Proof.** Refer to Lee[2]. ■

By using proposition 1, we can characterize every extreme point of *P* as follows.

**Proposition 2.** *Let $(\bar{x},\bar{z})\in P$ be an extreme point of P, then :*

$$\bar{x}_k=l_k/2,\quad \text{for all } k\in K,$$

*where $l_k$ is a nonnegative integer which is less than or equal to $2r_k$.*
**Proof.** It is clear from *i*) of proposition 1. ■

Now, we will analyze the strength of the bound of (LP). Let $z_{LP}$ be the optimal objective value of (LP) and let $z_{RLP}$ be the optimal objective value of (RLP).

**Proposition 3.** *Given optimal extreme point solution* $(x^*, z_{LP})$ *to (LP), we can construct a feasible solution* $(x', z')$ *to (RLP) such that* $z' \le \lfloor z_{LP} \rfloor + 1$.

**Proof.** Refer to Lee[2]. ■

**Theorem 1.** $z_{RLP} - z_{LP} \le 1$ *and the bound is tight.*

**Proof.** It is clear that $z_{RLP} - z_{LP} \le 1$ from proposition 3. Consider an instance of RLP defined on a 4-nodes cycle with only two commodities ( node 1 – node 3, node 2 – node 4) whose demands are all equal to 1. In this case, $z_{LP} = 1$ and $z_{RLP} = 1$. So, the bound is tight. ■

## 4. A Strengthened LP Formulation

For a pair of inequalities $I_e$ and $I_f$, if exactly one of $R_e$ and $R_f$ is an odd number, we can obtain the following valid inequality $I_{ef}$ to (RLP) :

$$I_{ef} : -z + \sum_{\{k\in K|e,f\in P_k^+\}} x_k - \sum_{\{k\in K|e,f\in P_k^-\}} x_k \le \left\lfloor \frac{(R_e + R_f)}{2} \right\rfloor,$$

where $L_{ef}(x,z) = (L_e(x,z) + L_f(x,z))/2$. Note that the right-hand-side of $I_{ef}$ is equal to $(R_e + R_f)/2 - 0.5$ and $I_{ef} = I_{fe}$.

For ease of exposition, let $Q = \{(e,f) \mid$ exactly one of $R_e$ and $R_f$ is odd, for each pair of $e, f \in E\}$. Let (LP') be the LP relaxation of (RLP) obtained by adding all $I_{ef}$, $(e,f) \in Q$ to (LP), which yields a stronger LP-relaxation of (RLP) than (LP). We also use $EQ(\bar{x}, \bar{z})$ to denote the set of inequalities of (LP') which are satisfied at equalities by a feasible solution $(\bar{x}, \bar{z})$ to (LP').

**Proposition 4.** *Let* $(\bar{x}, \bar{z})$ *be a feasible solution to* (LP'). *If* $I_e \in EQ(\bar{x}, \bar{z})$ *and* $I_f \in EQ(\bar{x}, \bar{z})$, *then both* $R_e$ *and* $R_f$ *are either odd or even.*

**Proof.** Suppose that $I_e \in EQ(\bar{x}, \bar{z})$ and $I_f \in EQ(\bar{x}, \bar{z})$, but exactly one of $R_e$ and $R_f$ is odd. Then,

$$L_{ef}(\bar{x}, \bar{z}) = (L_e(\bar{x}, \bar{z}) + L_f(\bar{x}, \bar{z}))/2 = \frac{(R_e + R_f)}{2} > \left\lfloor \frac{(R_e + R_f)}{2} \right\rfloor,$$

hence $(\bar{x}, \bar{z})$ violates $I_{ef}$. ■

**Proposition 5.** *Let* $(x^*, z^*)$ *be an extreme point solution to* (LP') *with* $z^* = z_{LP}$. *If* $I_{ef} \in EQ(x^*, z^*)$, *for some* $(e,f) \in Q$, *then exactly one of* $I_e$ *and* $I_f$ *is in* $EQ(x^*, z^*)$.

**Proof.** Suppose that $I_{ef} \in EQ(x^*, z^*)$. Clearly, both of $I_e$ and $I_f$ cannot be in $EQ(x^*, z^*)$. Now, suppose that $I_e \notin EQ(x^*, z^*)$ and $I_f \notin EQ(x^*, z^*)$. Then the followings hold :

$L_e(x^*, z^*) + s_e = R_e$, $L_f(x^*, z^*) + s_f = R_f$ and $L_{ef}(x^*, z^*) = \lfloor (R_e + R_f)/2 \rfloor$, where $s_e > 0$ and $s_f > 0$.

From the construction of $I_{ef}$, $L_e(x,z) = L_{ef}(x,z) + l(x)$ and $L_f(x,z) = L_{ef}(x,z) - l(x)$, where $l(x) = \sum_{\{k\in K|e\in P_k^+, f\in P_k^-\}} x_k - \sum_{\{k\in K|e\in P_k^-, f\in P_k^+\}} x_k$.

That is, $L_e(x^*, z^*) + s_e = L_{ef}(x^*, z^*) + l(x^*) + s_e = R_e$, $L_f(x^*, z^*) + s_f = L_{ef}(x^*, z^*) - l(x^*) + s_f = R_f$, $L_{ef}(x^*, z^*) + (s_e + s_f)/2 = (R_e + R_f)/2$.

Therefore, $s_e + s_f = 1$, where $s_e > 0$ and $s_f > 0$. We now prove a claim.

**Claim 1.** $I_g \notin EQ(x^*, z^*)$, *for all* $g \in E \setminus \{e,f\}$.

**Proof.** Suppose $I_g \in EQ(x^*, z^*)$, for some $g \in E \setminus \{e,f\}$. Without loss of generality, assume that $R_g$ is odd and $R_e$ is even, then, since $0 < s_e < 1$,

$$L_{eg}(x^*, z^*) = (L_e(x^*, z^*) + L_g(x^*, z^*))/2 = \frac{(R_e + R_g)}{2} - \frac{s_e}{2} > \frac{(R_e + R_g)}{2} - \frac{1}{2} = \left\lfloor \frac{(R_e + R_g)}{2} \right\rfloor$$

Therefore, $(x^*, z^*)$ violates $I_{eg}$. ♦

Note that, by the assumption, $(x^*, z^*)$ is also an optimal solution to (LP). By claim 1, $L_e(x^*, z^*) < R_e$, for all $e \in E$. Therefore, there exist a feasible solution $(x^*, z')$ to (LP) such that $z' < z^*$, which is a contradiction. ■

The following theorem characterizes every optimal extreme point solution to (LP').

**Theorem 2.** *If there exists a feasible solution to* (RLP) *whose objective value is equal to* $z_{LP}$*, then an optimal extreme point solution* $(\hat{x}, \hat{z})$ *to* (LP') *is integral with* $\hat{z} = z_{LP}$*. Otherwise,* $(\hat{x}, \hat{z})$ *has possibly fractional coordinates with* $\hat{z} > z_{LP}$.

**Proof.** Suppose that there exists a feasible solution to (RLP) whose objective value is equal to $z_{LP}$. Then, there exists an optimal extreme point solution $(\hat{x}, \hat{z})$ to (LP') with $\hat{z} = z_{LP}$. Therefore we have only to prove that $(\hat{x}, \hat{z})$ is integral. By Proposition 4, $R_e$'s have the same parity, for all $I_e \in EQ(\hat{x}, \hat{z})$. Also by Proposition 5, if $I_{ef} \in EQ(\hat{x}, \hat{z})$, exactly one of $I_e$ and $I_f$ is in $EQ(\hat{x}, \hat{z})$. Let us assume that $I_{ef} \in EQ(\hat{x}, \hat{z})$ and $I_e \in EQ(\hat{x}, \hat{z})$. Then, $L_f(\hat{x}, \hat{z}) = R_f - 1$. Also note that $I_{ef}$ is equal to an inequality $(I_e + I'_f)/2$, where $I'_f : L_f(x, z) \le R_f - 1$. Therefore, $(\hat{x}, \hat{z})$ should be the unique solution to the following system of linear equations :

$x_k = r_k$, for all $k \in K$ such that $\hat{x}_k = r_k$,
$x_k = 0$, for all $k \in K$ such that $\hat{x}_k = 0$,
$L_e(x, z) = R_e$, for all $I_e \in EQ(\hat{x}, \hat{z})$, (3)
$L_f(x, z) = R_f - 1$, for all $I_{ef} \in EQ(\hat{x}, \hat{z})$ and $I_e \in EQ(\hat{x}, \hat{z})$. (4)

Note that the right-hand-side values of all equations in (3) and (4) have the same parity. Now, consider some $k \in K$ such that $\hat{x}_k = r_k$. Note that $x_k$ appears in all equations in (3) and (4) with the coefficients 1 or -1. Therefore, the right-hand side values of all equations in (3) and (4) after substituting $x_k = r_k$ into them also have the same parity. By repeating the same process, finally, we can obtain a system of linear equations similar to (1) with the right-hand sides of the same parity. Now, by Proposition 1, $(\hat{x}, \hat{z})$ is integral. This completes the first part of this theorem.

Now, suppose that $z_{RLP} > z_{LP}$. Then $\hat{z} > z_{LP}$, otherwise, as in the first part of this proof, $(\hat{x}, \hat{z})$ is an integral feasible solution to (LP') with $\hat{z} = z_{LP}$, which contradicts $z_{RLP} > z_{LP}$. ■

From the above theorem, (LP') either gives an optimal integral solution to (RLP) if $z_{RLP} = z_{LP}$ or proves $z_{RLP} > z_{LP}$ when $z_{LP} = p$ for some nonnegative integer *p*. Moreover, since $z_{RLP} - z_{LP} < 1$, it is clear that $z_{RLP} - z_{LP'} < 1$, thus, $z_{RLP} = \lceil z_{LP'} \rceil$, where $z_{LP'}$ is the optimal objective value of (LP'). The number of inequalities $I_{ef}, (e, f) \in Q$, is at most $|V^2|/4$. Therefore, (LP') has $|K|$ variables and at most $|V| + |V^2|/4$ constraints.

## 5. Concluding Remarks

In this paper, we present a strengthened linear program, with size polynomially bounded by the input size, which provides enough information to determine the optimal value of (RLP). We think that it is an interesting and worthwhile subject to study the complete inequality description of the convex hull of feasible solutions of (RLP).

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