대한산업공학회/한국경영과학회 2004 춘계학술대회
2004년 2월 21 ~ 22일 전북대학교

AHP의 일치성 척도의 활용에 관한 연구
A Study on the Utilization of Compatibility Metric in the AHP

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Abstract

This study proposes two utilization ways of Saaty's compatibility metric to an entire hierarchy: (a) composite mode of all priorities and compatibility indices pertaining to a hierarchy, (b) arithmetic mean of compatibility indices along the hierarchy levels using a reduced elementwise operation of two eigenvectors.

1. Introduction

Concerning the Analytic Hierarchy Process (AHP), a group judgment has been one of its major issues [6]. In this context, Saaty[4] developed a compatibility metric of different results in the AHP. Saaty's compatibility index measures how close two pairwise reciprocal matrices of an order are. This is useful in applying the AHP to the delphic method.

However, any compatibility metric for an entire hierarchy has not yet been proposed. The AHP is known by taking advantages of hierarchic composition by breaking a problem down into its constituent parts and relating them in a logical fashion from the large, descending in gradual steps, to the smaller [7]. Thereby, compatibility tests are required for an entire hierarchy as well as for an pairwise comparison matrix.

This study is motivated to solve the problem and suggests two compatibility metrics for a hierarchy. One of the two is composite mode of priorities and compatibility indices. The other is recursive mode of eigenvectors, which will be illustrated.

2. Compatibility Metric in the AHP

The AHP is based on a matrix composed of ratio scale judgement and its eigenvector. In order to obtain priorities (or weights) for a set of n objects, the AHP begins with construction of a pairwise comparison matrix, $A=(a_{ij})$, where a_{ij} indicates the relative preference between the priority of ith object (w_i) and that of jth object (w_j). This matrix has positive entries everywhere and satisfies the reciprocal property $a_{ji}=1/a_{ij}$, which is called a reciprocal matrix.

In a real decision making environment, people's perceived relative preferences in pairwise comparisons remain inconsistent and intransitive, i.e., $a_{ij}\cdot a_{jk}\neq a_{ik}$. From the form $A\cdot w=\lambda_{\max}\cdot w$ $(\lambda_{\max}\geq n)$, where $\lambda_{\max}$ is the principal eigenvalue of A and w is a corresponding eigenvector, inconsistency in A can be captured by a single number $\lambda_{\max}-n$, which measures the deviation of the judgements from the consistent approximation and leads to consistency index (CI) as follows;

$$CI=\frac{\lambda_{\max}-n}{n-1} \qquad (1)$$

Random index (RI) is obtained as an average over CI values from large number of randomized reciprocal matrices (See Saaty[2], p. 21, for more details). The ratio between the two indices (CI/RI) is defined as consistency ratio (CR).

In addition to CI, Saaty[4] developed compatibility metric of different results in the AHP using the Hadamard product. Let $\odot$ be the Hadamard product in this paper and its multiplication is defined as $A \odot B = (a_{ij} \cdot b_{ij})$ for two matrices, $A=(a_{ij})$ and $B=(b_{ij})$. If A is a positive reciprocal matrix, $e^T A \odot A^T e = n^2$, where $e^T = (1, 1, \cdots, 1)$. If let $W_{nxn} = (w_i / w_j)$ be the matrix of ratios between paired elements of the principle right eigenvector, $w^T = (w_1, w_2, \cdots, w_n)$ and $\sum w_i = 1$, of the pairwise comparison matrix A, compatibility index (SI) is as follows;

$$SI = \frac{1}{n^2} e^T A \odot W^T e \quad (2)$$
$$\left(= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \frac{\lambda_{\max}}{n} \right)$$

SI can be used to measure the closeness of two same ordered matrices and it is proved that $e^T A \odot B^T e = n^2$, thereby, $SI = 1$ if and only if the two corresponding principle eigenvectors are exactly same. Matching SI with the acceptable level of $CI (= CR \times RI)$ through $\lambda_{\max}$ of (1) and (2), the significant level of $SI (= 1 + CI(n-1)/n)$ to make two matrices compatible is shown in Table 1 for each size of matrix.

Table 1. Consistency and Compatibility

Size (n)	CR	RI	CI	$\lambda_{\max}$	SI
3	0.05	0.52	0.026	3.052	1.017
4	0.08	0.89	0.071	4.214	1.053
5	0.10	1.11	0.111	5.444	1.089
6	0.10	1.25	0.125	6.625	1.104
7	0.10	1.35	0.135	7.810	1.116
8	0.10	1.40	0.140	8.980	1.123
9	0.10	1.45	0.145	10.160	1.129

3. Compatibility Metric for an Entire Hierarchy

3.1 Composite Mode

Saaty[2] generalized the metric of consistency ratio to an entire hierarchy (CRH) using the ratio between consistency index and random index of an hierarchy. The key idea is that all indices from pairwise comparisons pertaining to a hierarchy are aggregated to the whole according to the priority of each components in the hierarchy.

We apply the idea of CRH to the development of compatibility index for an entire hierarchy (let it be SIH). What we do is to multiply the index of compatibility between two pairwise reciprocal matrices by the priority of the objects with respect to which the comparison is made and add all the results for the entire hierarchy. This is then compared with the corresponding critical value of index ($SIcH$) obtained by taking acceptable level of compatibility (*See* Table 1.), weighting it by the priorities and adding. The ratio between the two indices ($DSIH$) is used to determine two sets of judgments of a hierarchy compatible. These are formulated as in (3).

$$DSIH = \frac{SIH}{SIcH} \quad (3)$$

where

$$\left. \begin{aligned} SIH &= P^0 \cdot \Psi^1 + (P^1)^T \Psi^2 + \cdots + (P^{H-1})^T \Psi^H \\ SIcH &= P^0 \cdot \Gamma^1 + (P^1)^T \Gamma^2 + \cdots + (P^{H-1})^T \Gamma^H \end{aligned} \right\}$$

$$(P^h)^T = \left(w(x_1^h), w(x_2^h), \cdot\cdot\cdot\cdot, w(x_{n_h}^h) \right),$$

$$(\Psi^{h+1})^T = \left(SI(x_1^h), SI(x_2^h), \cdot\cdot\cdot\cdot, SI(x_{n_h}^h) \right),$$

$$(\Gamma^{h+1})^T = \left(SIc(x_1^h), SIc(x_2^h), \cdot\cdot\cdot\cdot, SIc(x_{n_h}^h) \right),$$

$w(x_i^h)$: the priority of x_i^h,

$SI(x_i^h)$: the compatibility index of immediate lower level attributes of x_i^h,

$SIc(x_i^h)$: the critical value corresponding to $SI(x_i^h)$

x_i^h: the ith attribute in the hth level only that has immediate lower level attributes

H: the depth of the hierarchy

Each of P^0, Ψ^1, and Γ^1 in (3) is a scalar, and generally P^0 is equal to unit. If $DSIH$ is less than equal to 1, the two judgments for a hierarchy are compatible. Otherwise, those two are mutually different.

However, in applying this metric, we have to pay attention to priorities of $x_i^h s$ because two sides' priorities usually different. There are a few ways recommended; (a) geometric mean of pairwise comparisons, (b) elementwise arithmetic mean of eigenvectors, (c) elementwise geometric mean of eigenvectors, and (d) weighted geometric mean of pariwise comparisons. The weights in (d) depends on judgment capabilities of evaluators [1].

3.2 Recursive Mode

If two vectors $w^T = (w_1, w_2, \cdots, w_n)$ and $v^T = (v_1, v_2, \cdots, v_n)$ are principle right eigenvectors of positive reciprocal matrices A and B respectively, then we suggest that SI_{WV} be an alternative for SI_{AB}, where $W = (w_i/w_j)_{n \times n}$, $V = (v_i/v_j)_{n \times n}$ and

Under the condition that $CR \le 0.1$, the eigenvector of a given pairwise reciprocal matrix can reflect the decision maker's actual opinion of priorities [3]. Conceptually SI_{AB} starts with comparison of two matrices and arrives at determining the closenes of two eigenvectors derived respectively from each matrix. Whereas, SI_{WV} of this paper starts with driving two eigenvectors from the two conrresponding matrices and constructs two consistent pairwise reciprocal matrices to be followed by required operations for compatibility.

However, SI_{WV} is recursively reduced to simple operations of the elements of the two eigenvectors instead of operations of the matrices W and V. The formula and its derivation are as follows.

$$SI_{WV} = \frac{1}{n^2}\left(\sum_i \frac{v_i}{w_i}\right)\left(\sum_i \frac{w_i}{v_i}\right) \tag{4}$$

Proof)

Let $V = (\frac{v_i}{v_j})_{n \times n} = (\frac{w_i}{w_j}\delta_{ij})_{n \times n}$, where $\delta_{ij}(>0) = 1/\delta_{ji}$ is the incompatibility factor to the corresponding element of W. The compatibility index between the two matrices is described as follows;

$$SI_{WV} = \frac{1}{n^2} e^T W \odot V^T e$$

$$W \odot V^T = \left(\frac{w_i}{w_j}\frac{w_j}{w_i}\delta_{ji}\right)_{n \times n} = (\delta_{ji})_{n \times n}$$

$$\begin{aligned} e^T W \odot V^T e &= [\, n + \sum_{1 \le i < j \le n} (\delta_{ji} + \delta_{ij}) \,] \\ &= [\, n + \sum_{all\ i \ne j} (\delta_{ij}) \,] \end{aligned} \tag{5}$$

From $\delta_{ij} = \frac{v_i}{w_i} \cdot \frac{w_j}{v_j}$, (5) is rewritten as

$$\begin{aligned} e^T W \odot V^T e &= \left[n + \sum_{all\ i \ne j} \left(\frac{v_i}{w_i} \cdot \frac{w_j}{v_j}\right) \right] \\ &= \left[n + \sum_{all\ i} \left\{ \frac{v_i}{w_i} \left(\sum_{all\ j} \frac{w_j}{v_j}\right) - 1 \right\} \right] \end{aligned}$$

As a result,

$$e^T W \odot V^T e = \left(\sum_i \frac{v_i}{w_i}\right)\left(\sum_i \frac{w_i}{v_i}\right) \text{ and}$$

$$SI_{WV} = \frac{1}{n^2}\left(\sum_i \frac{v_i}{w_i}\right)\left(\sum_i \frac{w_i}{v_i}\right) \qquad \textit{Q.E.D.}$$

Using equation (4), we propose another compatibility index for an entire hierarchy (*MSIH*) as follows;

$$MSIH = \frac{\sum_h SI_{WV}^{(h)}}{H} \tag{6}$$

$SI_{WV}^{(h)}$: compatibility between matrix W and V in level h of the hierarchy, where W and V are composed of priorities of all elements in level h.
H: the depth of the hierarchy

In order to determine the two judgments for a hierarchy are compatible, *MSIH* should be less than or equal to 1.1. Recursive mode is the simpler of the two modes of this study.

3.3. Example

In order to apply proposed compatibility metrics, we adopt a appropriate case of software quality evaluation between beginner class and advanced class, which is cited from the results of Yoon's study[5]. The hierarchy is composed of two levels, quality characteristics and sub-characteristics, and its priority vector for each level is shown in Table 2. The data in those tables was obtained from the judgments of end users, classified into beginners and advanced, of packages. This paper illustrates the extent to which compatibility of the two classes' judgments for the software quality hierarchy.

Table 2 shows each compatibility index between two groups' judgments at each level. For example, the arithmetic operation is given below for SI_{WV} between priority vector of beginner class (w^T) and that of advanced class (v^T) for characteristics of level 1 in Table 2.

$w^T=$ (0.212, 0.162, 0.244, 0.242, 0.141),

$v^T=$ (0.340, 0.216, 0.194, 0.149, 0.101),

$$SI_{WV}^{(1)}=\frac{1}{5^2}\left(\sum_{i=1}^{5}\frac{w_i}{v_i}\right)\left(\sum_{i=1}^{5}\frac{v_i}{w_i}\right)=1.145.$$

SI_{WV} for 16 characteristics of level 2 obtained by the same way.

$$SI_{WV}^{(2)}=\frac{1}{16^2}\left(\sum_{i=1}^{16}\frac{w_i}{v_i}\right)\left(\sum_{i=1}^{16}\frac{v_i}{w_i}\right)=1.1940.$$

Finally *MSIH* = (1.1449 +1.1940)/2 = 1.1694. The result is greater than 1.1 and two groups' judgments for the entire hierarchy are not compatible.

Table 2. Compatibility for A Hierarchy

	Beginner(w)	Advanced(v)	(w/v)	(v/w)
F	0.2120	0.3404	0.6228	1.6057
R	0.1617	0.2159	0.7490	1.3352
U	0.2438	0.1936	1.2593	0.7941
E	0.2416	0.1487	1.6247	0.6155
P	0.1409	0.1014	1.3895	0.7197
SI of Level 1= 1.149			5.6453	5.0701
F1	0.0420(0.1981)	0.1149(0.3374)	0.3657	2.7347
F2	0.0499(0.2355)	0.0691(0.2030)	0.7225	1.3841
F3	0.0562(0.2649)	0.0888(0.2608)	0.6326	1.5808
F4	0.0237(0.1119)	0.0299(0.0879)	0.7928	1.2613
F5	0.0402(0.1896)	0.0378(0.1109)	1.0648	0.9392
R1	0.0483(0.2989)	0.0766(0.3546)	0.6313	1.5840
R2	0.0407(0.2516)	0.0506(0.2342)	0.8046	1.2429
R3	0.0727(0.4495)	0.0888(0.4111)	0.8189	1.2211
U1	0.0874(0.3583)	0.0571(0.2950)	1.5295	0.6538
U2	0.0636(0.2609)	0.0653(0.3375)	0.9735	1.0272
U3	0.0928(0.3808)	0.0711(0.3675)	1.3049	0.7664
E1	0.1259(0.5210)	0.0909(0.6113)	1.3847	0.7222
E2	0.1157(0.4790)	0.0578(0.3887)	2.0022	0.4995
P1	0.0701(0.4972)	0.0469(0.4621)	1.4951	0.6689
P2	0.0421(0.2985)	0.0341(0.3364)	1.2330	0.8110
P3	0.0288(0.2043)	0.0204(0.2015)	1.4089	0.7098
SI of Lever 2 = 1.1940			17.1650	17.8067
MSIH = (1.1449 +1.1940)/2 = **1.1694** $\geq$ 1.1				

4. Conclusion

This paper proposed and discussed compatibility metrics for an entire hierarchy in the AHP. The two metrics of this study are based on Saaty's compatibility. The first of the two is weighted average of compatibility indices along the hierarchic decomposition. The key idea is adopted from consistency index of a hierarchy. The second is originated from the fact that the actual judgment differences are priorities of eigenvectors. This paper derived a reduced formula of two eigenvectors' elements from two pairwise reciprocal matrices' operations. Then this study developed a compatibility metric for a hierarchy as the mean of compatibility indices along the hierarchy levels.

This study recommend the second metric should be practical because getting to mean priorities vector is hard and complex in the first metric of this study. This study will helpful to the utilization of compatibility index in the AHP.

References

[1] Cho, K.T., "Aggregation of Multiple Evaluator's Weights in Applying the AHP to Evaluate Technology Alternatives," *Korean Management Science Review* 19(2), pp. 139-153, 2002 (Korean).

[2] Saaty, T.L., *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.

[3] Saaty, T.L., "How to Make a Decision: The Analytic Hierarchy Process", *European Journal of Operational Research* 48, pp. 9-26, 1990.

[4] Saaty, T.L., *The Analytic Network Process*, RWS Pub., Pittsburgh, 1996, 2001.

[5] Yoon, M.S., *S/W Quality Evaluation Model Using the AHP - Developing a New Judgments Aggregation Method*, Thesis for The Degree of Doctor, Graduate School of Korea University, 1997.

[6] Yoon, M.S. and K.S. Whang, "Pooling Partial Pairwise Comparisons in the AHP", *International Journal of Management Science* 4, pp. 35-57, 1998.

[7] Vargas, L.G., "An Overview of the Analytical Hierarchy Process and Its Applications", *European Journal of Operational Research* 48, pp. 2-8, 1990.