Fourier Convergence Analysis Applied to Neutron Diffusion Eigenvalue Problem

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1. Introduction

Fourier error analysis has been a standard technique for the stability and convergence analysis of linear and nonlinear iterative methods [1,2,3,4]. Though the methods can be applied to eigenvalue problems too, all the Fourier convergence analyses have been performed only for fixed source problems and a Fourier convergence analysis for eigenvalue problem has never been reported.

Lee et al proposed new 2-D/1-D coupling methods and they showed that the new ones are unconditionally stable while one of the two existing ones is unstable at a small mesh size and that the new ones are better than the existing ones in terms of the convergence rate[4].

In this paper the convergence of method A in reference 4 for the diffusion eigenvalue problem was analyzed by the Fourier analysis. The Fourier convergence analysis presented in this paper is the first one applied to a neutronics eigenvalue problem to the best of our knowledge.

2. Fourier Analysis for an Eigenvalue Problem

2.1 A 2-D/1-D Coupling Method for an Eigenvalue Problem

The 2-D/1-D coupling methods described in reference 4 can be directly applied to eigenvalue problems. They begin with the axially averaged 2-D diffusion equation which can be written for each plane as in Eq. (1),

$$-\left(\frac{\partial}{\partial x}D_k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D_k\frac{\partial}{\partial y}\right)\bar{\phi}_k+\Sigma_k\bar{\phi}_k = \nu\Sigma_{f,k}\bar{\phi}_k/k_{eff}-\left(J_{z,k+1}-J_{z,k}\right)/h_{z,k} \quad (1)$$

and the radially averaged 1-D diffusion equation for each axial mesh which can be written as in Eq. (2),

$$-\frac{\partial}{\partial z}D_{i,j}\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma_{(i,j)}\phi_{(i,j)}=\nu\Sigma_{f,(i,j)}\phi_{(i,j)}/k_{eff}-\left(J_{x,i+1}-J_{x,i}\right)/h_x-\left(J_{y,j+1}-J_{y,j}\right)/h_y \quad (2)$$

Method A in reference 4 is to evaluate the TL of the 2-D equation directly from the 1-D solution. The effective multiplication factor can be updated by applying the power iteration as follow :

$$k_{eff}^{(n)}=k_{eff}^{(n-1)}\frac{\int_V w\nu\Sigma_f\phi^{(n)}dV}{\int_V w\nu\Sigma_f\phi^{(n-1)}dV}, \quad (3)$$

where w is an arbitrary weighting function.

2.2 Model Problem for the Convergence Analysis

A model problem was developed to analyze the convergence of method A applied to an eigenvalue problem. The model problem used here is a 3-D one-group diffusion eigenvalue problem in a homogeneous finite multiplying medium of N planes with periodic boundary conditions. It is obvious that the exact solution to the model problem is $\phi=\phi_0$ (arbitrary constant) and $k_{eff}=k_\infty=\nu\Sigma_f/\Sigma$. Two basic assumptions are introduced in order to simplify the convergence analysis. These are (1) solving the 2-D problems plane by plane, which means solving them iteratively in the z-direction and (2) solving the 2-D problem by a direct inversion of the 2-D operator in a given plane. The second assumption leads to a zero radial leakage during the iterations, and simplifies Eqs (1) and (2) to:

$$\Sigma\bar{\phi}_k=\nu\Sigma\bar{\phi}_k/k_{eff}-\left(J_{z,k+1}-J_{z,k}\right)/h, \quad (4a)$$

$$-\frac{\partial}{\partial z}D\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma\phi_{(i,j)}=\frac{1}{k_{eff}}\nu\Sigma\phi_{(i,j)}. \quad (4b)$$

2.3 Fourier Convergence Analysis of Method A

The iterative algorithm of method A applied to the eigenvalue problem with one inner iteration per outer iteration can be expressed by the following equations :

$$\Sigma\bar{\phi}_k^{(n)}=\frac{1}{k_{eff}^{(n-1)}}\nu\Sigma_f\bar{\phi}_k^{(n-1)}-\frac{1}{h}\left(J_{z,k+1}^{(n-1)}-J_{z,k}^{(n-1)}\right), \quad (5a)$$

$$k_{eff}^{(n)}=k_{eff}^{(n-1)}\sum_{k'}\bar{\phi}_{k'}^{(n)}\Big/\sum_{k'}\bar{\phi}_{k'}^{(n-1)}, \quad (5b)$$

$$J_{z,k}^{(n)}=-A^{(n)}\left(\bar{\phi}_k^{(n)}-\bar{\phi}_{k-1}^{(n)}\right), \quad (5c)$$

where

$$A^{(n)}=\frac{D\left(\kappa^{(n)}\right)^2h}{4\sinh^2\left[\kappa^{(n)}h/2\right]};\quad \kappa^{(n)}=\sqrt{\frac{\Sigma-\nu\Sigma_f/k_{eff}^{(n)}}{D}}$$

Note that the two-node analytic nodal method was used to solve the axial 1-D equation. The node average fluxes on each plane and the continuity condition of the flux and the net current at the interface of the planes are used as the constraints for the axial two-node problem. And also note that a constant weighting function was used to get Eq. (5b).

By summing the Eq. (5a) for $k=0,1,\cdots,N-1$, we get the the following equation :

$$k_{eff}^{(n)}=k_\infty \quad (n>1), \quad (6)$$

which simplifies the Fourier analysis of method A applied to the eigenvalue problem.

As we did in the fixed source problem, let's introduce 66a first order perturbation of the flux. We also introduce a first order perturbation of $A^{(n)}$ in Eq. (5c) because it also depends on the iteration step.

$$\bar{\phi}_k^{(n)}=\phi_0\left(1+\varepsilon\xi_k^{(n)}\right), \quad (7a)$$

$$A^{(n)}=\left(D/h\right)\left(1+\varepsilon\theta^{(n)}\right). \quad (7c)$$

Inserting Eq. (7) into Eq. (5) and dropping the $O(\varepsilon^2)$ terms yields the following linearized equation :

$$\xi_k^{(n)} = \xi_k^{(n-1)} + \frac{L^2}{h^2}\left(\xi_{k-1}^{(n-1)} - 2\xi_k^{(n-1)} + \xi_{k+1}^{(n-1)}\right), \quad (8)$$

By inserting the following Fourier ansatz into Eq. (8) :

$$\xi_k^{(n)} = a\omega^n e^{i\lambda(k+1/2)h}, \quad (9)$$

we obtain the following equation :

$$\omega = 1 + 2\frac{L^2}{h^2}\left[\cos(\tau) - 1\right]; \quad \tau = \lambda h. \quad (10)$$

Note that unlike the fixed source problem, only some discrete values of λ are allowed in Eq. (9). There are only N independent basis for the flux because the dimension of the flux vector is N. We can choose the N eigenvectors from the lowest mode as the basis. The flux can be expanded by the N eigenvectors corresponding to the eigenmodes $\lambda_m = 2m\pi/(Nh)\ (m = 0,1,\cdots N-1)$ which satisfy the periodic boundary conditions of the model problem. Among the eigenmodes, $\lambda_0 = 0$ forms the fundamental mode solution of the flux, 1 for this model problem, and the other modes, $\lambda_m \quad (m = 1,2,\cdots N-1)$, form the error term of the flux, $\xi_k^{(n)}$. Note that only $N-1$ discrete values, $\tau_m = \lambda_m h \quad (m = 1,2,\cdots N-1)$, are allowed for τ in Eq. (10). They are $2\pi/N$, $4\pi/N$, $\cdots$, and $2(N-1)\pi/N$. The spectral radius of the linearized algorithm of method A for the eigenvalue problem is given by :

$$\rho = \underset{\tau=\tau_1\cdots\tau_{N-1}}{\text{Max}} |\omega| \, . \quad (11)$$

3. Results and Discussions

Figure 1 shows the spectral radius of method A as a function of the axial mesh size for the model problem with $N = 5$, $D = 0.833333, \Sigma = 0.02$, and $\nu\Sigma_f = 0.019$. The line is the analytic spectral radius obtained by the Fourier analysis and the dots are the numerical ones. As indicated, a good agreement is observed between the analytic and numerical results. The algorithm diverges at a small mesh size in the eigenvalue problem as it did in the fixed source problem. It is interesting that the spectral radius in the eigenvalue problem approaches 1 as the mesh size increases while it approaches zero in the fixed source problem. It was assumed that N is infinite in fixed source problem. However, One can also show that the spectral radius depends on N and the minimum value of it approaches 1 as N increases.

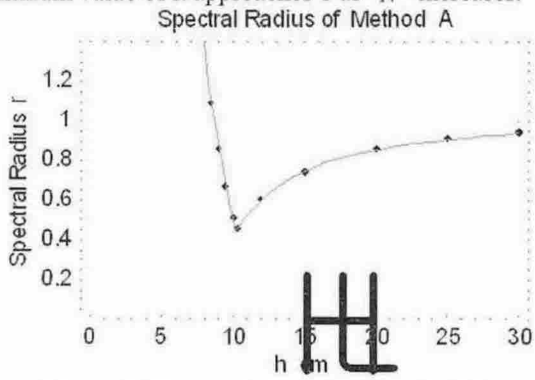

Figure 1. The spectral radius of method A

4. Conclusion

In this paper the convergence of method A in reference 4 applied to a diffusion eigenvalue problem was analyzed by the Fourier analysis. The Fourier convergence analysis presented in this paper is the first one applied to a neutronics eigenvalue problem to the best of our knowledge. The convergence behavior of method A in the eigenvalue problem is very different from that in the fixed source problem.

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