Suboptimal Adaptive Filters for Stochastic Systems with Multisensor Environment

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Abstract: An optimal combination of arbitrary number correlated estimates is derived. In particular, for two estimates this combination represents the well-known Millman and Bar-Shalom-Campo formulae for uncorrelated and correlated estimation errors, respectively. This new result is applied to the various estimation problems as least-squares estimation, Kalman filtering, and adaptive filtering. The new approximate adaptive filter with a parallel structure is proposed. It is shown that this filter is very effective for multisensor systems containing different types of sensors. Examples demonstrating the accuracy of the proposed filter are given.

Keywords: Kalman filtering, adaptive filtering, suboptimal filtering, least-squares estimator, Millman's formula, multisensor, data fusion

1. INTRODUCTION

In recent years, there has been growing interest to fuse multisensory data to increase the accuracy of estimation parameters and system states. This interest is motivated by the availability of different types of sensor which uses various characteristics of the optical, infrared, and electromagnetic spectrums. The measurements used in the estimation process are assigned to a common target as a result of the association process. different types of sensors ? The well-known Millman and Bar-Shalom-Campo formulae for two uncorrelated and correlated estimation errors, respectively, are widely used in the filtering and the smoothing problems [1-2]. But there is a need to generalize these formulae for the multisensor environment so that we can fuse more than two arbitrary dependent estimates.

The main purpose of this paper is a generalization of the Millman and Bar-Shalom-Campo formulae to arbitrary number of estimates, which we called generalized Millman's formula (GMF). And second purpose is to show how to apply the GMF in the several adaptive filtering problems.

2. THE GENERALIZED MILLMAN'S FORMULA

Suppose we have N unbiased estimates $\hat{x}_1,\ldots,\hat{x}_N$ of an unknown random vector $x \in \mathbf{R}^n$ with the associated error covariances $P_{ij} = \mathrm{cov}\{\tilde{x}_i, \tilde{x}_j\}$, $\tilde{x}_i = x - \hat{x}_i$. It is desired to find the overall linear estimate of x, that is, the optimal estimate of the form

$$\hat{x} = \sum_{i=1}^{N} c_i \hat{x}_i , \qquad \sum_{i=1}^{N} c_i = I_n , \tag{1}$$

where I_n is the $n \times n$ unit matrix, and $c_1,\ldots,c_N$ are $n \times n$ constant weighting matrices determined from the mean-square criterion,

$$J(c_1,\ldots,c_N) = E\left(\left\| x - \sum_{i=1}^{N} c_i \hat{x}_i \right\|^2 \right) \to \min_{c_i} \tag{2}$$

The following linear equations for unknown matrices $c_1,\ldots,c_N$ give a solution of this problem:

$$\sum_{i=1}^{N-1} c_i \left(P_{ij} - P_{iN} \right) + c_N \left(P_{Nj} - P_{NN} \right) = 0 ,$$

$$j = 1,\ldots,N-1, \quad \sum_{i=1}^{N} c_i = I_n . \tag{3}$$

Proof. Using Eq. (1), criterion (2) can be rewritten as follows:

$$J = \mathrm{tr}\left\{ E\left[\sum_{i,j=1}^{N} c_i (x - x_i)(x - \hat{x}_j)^T c_j^T \right] \right\}$$

$$= \mathrm{tr}\left[\sum_{i,j=1}^{N} c_i P_{ij} c_j^T \right] \to \min_{c_1,\ldots,c_N} . \tag{4}$$

Substituting the expression

$$c_N = I_n - \left(c_1 + \quad + c_{N-1} \right)$$

into Eq. (4), we obtain

$$\begin{aligned} J = \mathrm{tr}\Bigg\{ & \sum_{i,j=1}^{N-1} c_i P_{ij} c_j^T + \sum_{i=1}^{N-1} \left(c_i P_{iN} - P_{Ni} c_i^T \right) \\ & + \sum_{i,j=1}^{N-1} \left(c_i P_{iN} c_j^T + c_j P_{Ni} c_i^T \right) + P_{NN} \\ & - \left(\sum_{i=1}^{N-1} c_i \right) P_{NN} - P_{NN} \left(\sum_{j=1}^{N-1} c_j^T \right) \\ & + \sum_{i,j=1}^{N-1} c_i P_{NN} c_j^T \Bigg\} \to \min_{c_1,\ldots,c_{N-1}} . \end{aligned} \tag{5}$$

Next, use the formulae

$$P_{ji} = P_{ij}^T, \quad P_{ii} = P_{ii}^T, \quad i, j = 1, \ldots, N,$$

$$\frac{\partial}{\partial c_i}\left[\mathrm{tr}(c_i P)\right] = P^T, \quad \frac{\partial}{\partial c_i}\left[\mathrm{tr}\left(P c_i^T\right)\right] = P,$$

$$\frac{\partial}{\partial c_i}\left[\mathrm{tr}\left(c_i P c_i^T\right)\right] = c_i\left(P^T + P\right). \tag{6}$$

Let us differentiate each summand of the function (5) with respect to $c_1, \ldots, c_{N-1}$ using Eqs. (6) , and then set the result to zero, i.e.,

$$\frac{\partial J}{\partial c_i} = 0, \quad i = 1, \ldots, N-1$$

we have the linear algebraic equations (3) for the unknown weighting matrices $c_1, \ldots, c_N$.

This complete the derivation of the Eqs. (3).

3. ADAPTIVE FILTERING IN LINEAR SYSTEMS

We also consider the problem of recursive filtering for dynamic systems with unknown parameters. A new suboptimal unbiased filter based on GMF (1), (3) is herein proposed. This filter also can be applied for dynamic systems with multisensor environment to fuse local sensor's estimates to get more accurate estimates. The equation for error covariance characterizing the mean-square accuracy of the filter is derived.

Let's consider the following discrete-time dynamic system with unknown parameters in observation model:

$$x_{k+1} = F_k x_k + G_k v_k, \quad k \ge 0,$$

$$y_k = H_k(\theta) x_k + w_k, \tag{7}$$

where state $x_k \in \mathbf{R}^n$, measurement $y_k \in \mathbf{R}^m$, noises $v_k \sim N(0, Q_k)$ and $w_k \sim N(0, R_k(\theta))$; and $H_k(\theta)$ and $R_k(\theta)$ are matrices depending on the unknown parameter $\theta \in \mathbf{R}^p$, which takes only a finite set of values

$$\theta = \theta_1, \theta_2, \ldots, \theta_N .$$

In adaptive filtering theory, two filters are primarily used for estimation of the state vector. Both of these filters are based on the Bayesian approach in which the unknown parameter θ is assumed to be random with a *prior* known distribution [3-6]:

$$p(\theta) = p(\theta_1), \ldots, p(\theta_N),$$

$$p(\theta_1) + \quad + p(\theta_N) = 1, \quad p(\theta_i) \ge 0. \tag{8}$$

In the first filter, θ is treated as a random constant vector such as

$\theta_{k+1} = \theta_k$ or more efficiency,

$$\theta_{k+1} = \theta_k + \xi_k, \tag{9}$$

where $\{\xi_k\}$ is any zero-mean Gaussian white noise sequence. And the system (7) together with assumption (4) can be reformulated as the nonlinear model for the composite state vector $[x_k \quad \theta_k]^T$:

$$\begin{bmatrix} x_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} F_k x_k \\ H_k(\theta_k) x_k \end{bmatrix} + \begin{bmatrix} G_k v_k \\ \xi_k \end{bmatrix},$$

$$y_k = H_k(\theta_k) x_k + w_k,$$

and the suboptimal nonlinear filtering procedures (extended Kalman filter, and so on [1,3,6]) can be applied to estimate the augment state vector which contains θ_k as its components. It has been observed that the suboptimal nonlinear filters may give biased estimates and sometimes diverge. Also these filters are rather difficult to implement real-time, especially for multidimensional dynamic systems.

The second filter is based on the Lainiotis partitioning approach [4-5], which is separate the filtering process x_k from the identification of the unknown parameter θ. In this case the optimal mean square estimate $\hat{x}_k = E\left(x_k \middle| y^k\right)$ of the state x_k and the corresponding estimation error covariance $P_k = E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \middle| y^k\right]$ are given by the Lainiotis-Kalman filter equations:

$$\hat{x}_k = \sum_{i=1}^{N} p\left(\theta_i \middle| y^k\right) \hat{x}_k(\theta_i),$$

$$P_k = \sum_{i=1}^{N} p\left(\theta_i \middle| y^k\right)\{P_k(\theta_k) + [\hat{x}_k(\theta_i) - \hat{x}_k][\hat{x}_k(\theta_i) - \hat{x}_k]^T\}, \tag{10}$$

where

$$\hat{x}_k(\theta_i) = E\left(x_k \middle| y^k, \theta_i\right),$$

$$P_k(\theta_i) = E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \middle| y^k, \theta_i\right] \tag{11}$$

are the state estimate and its error covariance, respectively, given by the elemental discrete Kalman filter (KF) matched to the linear system (7) at fixed $\theta = \theta_i$, and $p(\theta_i | y^k)$ is *a posteriori* probability of θ_i given $y^k = \{y_1, \dots, y_k\}$, providing by Bayesian rule (see [4-5]). The filter (10), (11) yields an effective estimation algorithm only for low dimension of the parameter vector $\theta \in \mathbf{R}^p$, since it requires an evaluation of the conditional probability densities $p(\theta_i | y^k)$ at each time instance $k = 1,2,\dots$.

We propose an alternative adaptive filtering algorithm without the additional assumption (8) about *a prior* probability $p(\theta)$ of the parameter θ. This algorithm has no need of conditional densities $p(\theta_i | y^k)$ calculations. The new filter can be derived by using the standard Kalman filter (KF) for the system model (7) at the fixed value of the parameter $\theta = \theta_i$ ($i = 1,\dots,N$), and then combining the obtained kalman estimates $\hat{x}_k(\theta_1), \dots, \hat{x}_k(\theta_N)$ by using GMF (1), (3). The resulting filter will clearly be an approximate (suboptimal).

According to (7), we have N unconnected dynamic systems with known matrices $F_k, G_k, Q_k, H_k(\theta_i)$, and $R_k(\theta_i)$, respectively:

$$x_{k+1} = F_k x_k + G_k v_k, \quad v_k \sim N(0, Q_k),$$

$$y_k = H_k(\theta_i) x_k + w_k, \quad w_k \sim N(0, R_k(\theta_i)), \qquad (12)$$

where "**i**" is fixed and θ_i is the known value of the parameter θ. Using the KF matched to (12) at fixed θ_i, we have N estimates

$$\hat{x}_k(\theta_1), \dots, \hat{x}_k(\theta_N)$$

and associated error covariances

$$P_k(\theta_1), \dots, P_k(\theta_N).$$

Next, based on the GMF (1) the new adaptive estimate of the state x_k can be determined by the following equations:

$$\hat{x}_k^{GMF} = \sum_{i=1}^{N} c_k^{(i)} \hat{x}_k(\theta_i), \quad \sum_{i=1}^{N} c_k^{(i)} = I_n, \qquad (13)$$

where the weighting matrices $c_k^{(1)}, \dots, c_k^{(N)}$ determined by the GMF Eqs. (3) and KF Eqs.

Remark 1: Since θ takes a finite number of values $\theta = \theta_1, \theta_2, \dots, \theta_N$ the kalman estimates $\hat{x}_k(\theta_i)$ are separated for values of $i = 1,\dots,N$. Each estimate $\hat{x}_k(\theta_i)$ is found independently of other estimates $\hat{x}_k(\theta_1), \dots, \hat{x}_k(\theta_{i-1})$, $\hat{x}_k(\theta_{i+1}), \dots, \hat{x}_k(\theta_N)$. Therefore, it can be evaluated in parallel. The proposed filtering algorithm is also robust, since it can be corrected even if one of the parallel kalman estimate $\hat{x}_k(\theta_i)$ diverges. In this case, the corresponding weight matrix $c_k^{(i)}$ in the weighting sum (13) will tend to zero, thereby indicating that the diverging estimate $\hat{x}_k(\theta_i)$ will be discarded in the weighting sum of the (13).

Remark 2: We may note, that the kalman filter gains, the error covariances, and the weights $c_k^{(i)}$ may be precomputed, since they do not depend on the present observations y^k, but only on the noises statistics Q_k and $R_k(\theta_i)$, the system matrices $F_k, G_k, H_k(\theta_i)$, and also the values of parameter $\theta = \theta_1, \theta_2, \dots, \theta_N$, which are the part of system model. Thus, once the observation schedule has been settled, the real-time implementation of the proposed filter requires only the computation of the "local" kalman estimates $\hat{x}_k(\theta_1), \dots, \hat{x}_k(\theta_N)$ and the final estimate $\hat{x}_k^{GMF}$.

The proposed filter can be also applied for multisensor systems containing different types of sensors.

4. DATA FUSION OF MULTISENSOR's ESTIMATES

Consider a discrete-time linear dynamic system described by a difference equation with additive white Gaussian noise,

$$x_{k+1} = F_k x_k + G_k v_k, \quad k = 0,1,\dots, \qquad (14)$$

where $v_k \sim N(0, Q_k)$.

Suppose that multiple sensor (measurement system) involves N sensors,

$$\begin{aligned} y_k^{(1)} &= H_k^{(1)} x_k + w_k^{(1)}, \quad y_k^{(1)} \in \mathbf{R}^{m_1} \\ &\mp \\ y_k^{(N)} &= H_k^{(N)} x_k + w_k^{(N)}, \quad y_k^{(N)} \in \mathbf{R}^{m_N} \end{aligned} \qquad (15)$$

with $\{w_k^{(1)}\}, \dots, \{w_k^{(N)}\}$ are the sequences of zero-mean white Gaussian process noise, $w_k^{(i)} \sim \mathbf{N}(0, R_k^{(i)})$, $i = 1,\dots,N$. The initial state is modeled as a Gaussian random vector with known mean and covariance, $x_0 \sim N(\bar{x}_0, P_0)$. The N+1 noise sequences $\{v_k\}$, $\{w_k^{(i)}\}$, $i = 1,\dots,N$, and the initial state x_0 are mutually independent.

It is well-known that Kalman filter (KF) can be used to produce the optimal state estimate based on the results of overall measurements

$$Y_k = \{y_k^{(1)} \quad \dots \quad y_k^{(N)}\}, \quad Y_k \in \mathbf{R}^m,$$

$$m = m_1 + \quad + m_N. \qquad (16)$$

However, the computational cost and the numerical errors of the KF increase drastically with the state and measurement dimensions, for instance, in multisensor intelligent systems [7]. Hence, the KF may be impractical to implement. In such cases, reduced-order suboptimal filters are preferable since there is no need to estimate those states by using overall measurements Y_k simultaneously. In this paper, we show that the GMF may serve as an alternative to solve this problem.

The derivation of new suboptimal reduced-order filter is based on the assumption that the overall measurement vector Y_k consists of the combination of the different subvectors $y_k^{(1)},\ldots,y_k^{(N)}$, which can be processing separately. According to Eqs. (14), and (15), we have N unconnected dynamic subsystems ($i=1,\ldots,N$) with state vector $x_k \in \mathbf{R}^{\mathbf{n}}$ and measurement subvector $y_k^{(i)} \in \mathbf{R}^{\mathbf{m_i}}$:

$$x_{k+1} = F_k x_k + G_k v_k ,$$

$$y_k^{(i)} = H_k^{(i)} x_k + w_k^{(i)} , \qquad (17)$$

where $\mathbf{i}$ (the number of subsystem) is fixed.

Next, let us denote the estimate of the state x_k based on the measurement $y_k^{(i)}$ by $\hat{x}_{k|k}^{(i)}$. To find $\hat{x}_{k|k}^{(i)}$ we apply the KF to the subsystem (17). We have

$$\hat{x}_{k|k-1}^{(i)} = F_{k-1}\hat{x}_{k-1|k-1}^{(i)} , \quad \hat{x}_{0|0}^{(i)} = \bar{x}_0 ,$$

$$\hat{x}_{k|k}^{(i)} = \hat{x}_{k|k-1}^{(i)} + K_k^{(i)}\left[y_k^{(i)} - H_k^{(i)}\hat{x}_{k|k}^{(i)}\right],$$

$$P_{k|k-1}^{(i)} = F_{k-1}P_{k-1|k-1}^{(i)}F_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T , \quad P_{0|0}^{(i)} = P_0 ,$$

$$K_k^{(i)} = P_{k|k-1}^{(i)}\left(H_k^{(i)}\right)^T\left[\left(H_k^{(i)}\right)^T P_{k|k-1}^{(i)} H_k^{(i)} + R_k^{(i)}\right]^{-1} ,$$

$$P_{k|k}^{(i)} = \left[I_n - K_k^{(i)}H_k^{(i)}\right]P_{k|k-1}^{(i)} . \qquad (18)$$

In the Eqs. (18), $P_{k|k}^{(i)}$ is the filtering error covariance,

$$P_{k|k}^{(i)} = \mathrm{cov}\left\{\tilde{x}_{k|k}^{(i)}\right\}, \quad \tilde{x}_{k|k}^{(i)} = x_k - \hat{x}_{k|k}^{(i)} . \qquad (19)$$

Thus, from the equations (18) we have N partial filtering estimates

$$\hat{x}_{k|k}^{(1)} , \quad \ldots, \quad \hat{x}_{k|k}^{(N)} \qquad (20)$$

based on the measurements $y_k^{(1)}$, …, $y_k^{(N)}$, respectively, and corresponding error covariances

$$P_{k|k}^{(1)} , \quad \ldots, \quad P_{k|k}^{(N)} . \qquad (21)$$

Then the new suboptimal estimate $\hat{x}_{k|k}^{GMF}$ of the state vector x_k based on the overall measurements Y_k (16) is constructed from the partial estimates (20) by using the GMF (1):

$$\hat{x}_{k|k}^{GMF} = \sum_{i=1}^{N} c_k^{(i)}\hat{x}_{k|k}^{(i)} , \quad \sum_{i=1}^{N} c_k^{(i)} = I_n , \qquad (22)$$

where the time-varying weighting matrices $c_k^{(1)},\ldots,c_k^{(N)}$ determined by the Eqs. (3):

$$\sum_{i=1}^{N-1} c_k^{(i)}\left[P_{k|k}^{(ij)} - P_{k|k}^{(iN)}\right] + c_k^{(N)}\left[P_{k|k}^{(Nj)} - P_{k|k}^{(NN)}\right] = 0 ,$$

$$j = 1,\ldots, N-1, \quad \sum_{i=1}^{N} c_k^{(i)} = I_n , \qquad (23)$$

where

$$P_{k|k}^{(ii)} \overset{\Delta}{=} P_{k|k}^{(i)}$$

is the covariance (21) determined by the KF (18), and $P_{k|k}^{(ij)}$, $i \neq j$ is cross-covariance,

$$P_{k|k}^{(ij)} = \mathrm{cov}\left\{\tilde{x}_{k|k}^{(i)}, \tilde{x}_{k|k}^{(j)}\right\},$$

which satisfy the following recursion:

$$P_{k|k}^{(ij)} = \left[I_n - K_k^{(i)}H_k^{(i)}\right]\left[F_{k-1}P_{k-1|k-1}^{(ij)}F_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T\right] \times \left[I_n - K_k^{(j)}H_k^{(j)}\right]^T ,$$

$$P_{0|0}^{(ij)} = P_0 , \quad i \neq j, \quad i, j = 1,\ldots, N , \qquad (24)$$

where the gain $K_k^{(i)}$ is determined by the KF (18).

Thus, the KF (18), the GMF (22), (23), and the Eq. (24) completely define the new suboptimal multisensor filter. Note, that in this filter the estimates $\hat{x}_{k|k}^{(i)}$ are separated for different types of sensors. Therefore, the kalman filters (18) can be implemented in parallel for various values of $i = 1,\ldots, N$.

5. EXAMPLE: Identification of a Scalar Unknown Parameter

To estimate the value of a scalar unknown θ from two types of measurements corrupted by additive white Gaussian noises, the system and measurement models are

$$x_{k+1} = x_k, \quad x_k \equiv \theta,$$

$$y_k^{(1)} = x_k + w_k^{(1)}, \; y_k^{(2)} = x_k + w_k^{(2)}, \tag{25}$$

where

$$w_k^{(i)} \sim \mathbf{N}(0, r_i), \;\; i = 1,2; \;\; x_0 \sim \mathbf{N}(\bar{\theta}, \sigma_\theta^2).$$

The KF gives the optimal mean-square estimate $\hat{x}_k^{KF}$ of an unknown $x_k \equiv \theta$ based on the overall measurements

$$Y_k = \begin{bmatrix} y_k^{(1)} & y_k^{(2)} \end{bmatrix}^T.$$

In this case

$\hat{x}_{k|k-1}^{KF} = \hat{x}_{k-1|k-1}^{KF}$, and $P_{k|k-1}^{KF} = P_{k-1|k-1}^{KF}$

and the KF Eqs. (18) take the form:

$$\hat{x}_k^{KF} = \hat{x}_{k-1}^{KF} + K_k^{KF} \begin{bmatrix} y_k^{(1)} - \hat{x}_{k-1}^{KF} \\ y_k^{(2)} - \hat{x}_{k-1}^{KF} \end{bmatrix}, \quad \hat{x}_0^{KF} = \bar{\theta},$$

$$P_k^{KF} = \frac{r_1 r_2 P_{k-1}^{KF}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{KF}}, \quad P_0^{KF} = \sigma_\theta^2,$$

$$K_k^{KF} = \begin{bmatrix} \dfrac{r_2 P_{k-1}^{KF}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{KF}} \\ \dfrac{r_1 P_{k-1}^{KF}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{KF}} \end{bmatrix}^T, \tag{26}$$

For simplicity in Eqs. (26) we denote

$\hat{x}_k^{KF} \equiv \hat{x}_{k|k}^{KF}$, and $P_k^{KF} \equiv P_{k|k}^{KF}$.

Using the "step-by-step" induction, we obtain the exact formula for the mean square error P_k^{KF},

$$P_k^{KF} = E\left(\theta - \hat{x}_k^{KF}\right)^2 = \frac{\sigma_\theta^2}{1 + k r_{12} \sigma_\theta^2},$$

$$r_{12} = \frac{r_1 + r_2}{r_1 r_2}. \tag{27}$$

Together with the optimal KF (26), we apply the proposed adaptive filter based on the GMF. Let denote the partial estimates of the unknown $x_k \equiv \theta$ based on the single measurements $y_k^{(1)}$ and $y_k^{(2)}$ by $\hat{x}_k^{(1)}$ and $\hat{x}_k^{(2)}$, respectively. Using the system model with state x_k and single measurement $y_k^{(i)}$,

$$x_{k+1} = x_k, \;\; y_k^{(i)} = x_k + w_k^{(i)}$$

for $i = 1,2$, we obtain the equations for $\hat{x}_k^{(1)}$ and $\hat{x}_k^{(2)}$,

$$\hat{x}_k^{(i)} = \hat{x}_{k-1}^{(i)} + K_k^{(i)} \left[y_k^{(i)} - \hat{x}_{k-1}^{(i)} \right], \quad \hat{x}_0^{(i)} = \bar{\theta}.$$

$$K_k^{(i)} = \frac{P_{k-1}^{(i)}}{r_i + P_{k-1}^{(i)}}, \quad P_k^{(i)} = \left[1 - K_k^{(i)}\right] P_{k-1}^{(i)},$$

$$P_0^{(i)} = \sigma_\theta^2, \;\; i = 1,2. \tag{28}$$

The exact solutions of the Eqs. (28) take the form:

$$P_k^{(i)} = E\left(\theta - \hat{x}_k^{(i)}\right)^2 = \frac{r_i \sigma_\theta^2}{r_i + k \sigma_\theta^2},$$

$$K_k^{(i)} = \frac{r_i \sigma_\theta^2}{r_i + k \sigma_\theta^2}, \quad i = 1,2. \tag{29}$$

Next, using the GMF (1), (3) at $N = 2$, one can obtain the estimate $\hat{x}_k^{GMF}$ of the unknown $x_k \equiv \theta$ as

$$\hat{x}_k^{GMF} = c_k^{(1)} \hat{x}_k^{(1)} + c_k^{(2)} \hat{x}_k^{(2)},$$

$$c_k^{(1)} = \frac{P_k^{(2)} - P_k^{(12)}}{P_k^{(1)} - 2P_k^{(12)} + P_k^{(2)}},$$

$$c_k^{(2)} = \frac{P_k^{(1)} - P_k^{(12)}}{P_k^{(1)} - 2P_k^{(12)} + P_k^{(2)}}, \tag{30}$$

where the error variances

$$P_k^{(1)} \stackrel{\Delta}{=} P_{k|k}^{(11)} \quad \text{and} \quad P_k^{(2)} \stackrel{\Delta}{=} P_{k|k}^{(22)}$$

are determined by the formulae (29), and the cross-covariance

$$P_k^{(12)} \stackrel{\Delta}{=} P_{k|k}^{(12)},$$

according to the Eq. (24) is determined by the equation

$$P_k^{(12)} = \left[1 - K_k^{(1)}\right]\left[1 - K_k^{(2)}\right]P_{k-1}^{(12)}, \ P_0^{(12)} = \sigma_\theta^2. \qquad (31)$$

Using (29)-(31), one can obtain the exact expressions for $c_k^{(1)}$, $c_k^{(2)}$, and $P_k^{(12)}$, respectively. We have

$$c_k^{(1)} = \frac{r_2}{r_1 + r_2}, \quad c_k^{(2)} = \frac{r_1}{r_1 + r_2},$$

$$P_k^{(12)} = \frac{r_1 r_2 \sigma_\theta^2}{\left(r_1 + k\sigma_\theta^2\right)\left(r_2 + k\sigma_\theta^2\right)}. \qquad (32)$$

At last, using (29), (32), and (4) one can has the error variance of the suboptimal estimate $\hat{x}_k^{GMF}$,

$$P_k^{GMF} \stackrel{\Delta}{=} E\left(\theta - \hat{x}_k^{GMF}\right)^2,$$

$$P_k^{GMF} = \sum_{i,j=1}^{2} c_k^{(i)} P_k^{(ij)} c_k^{(j)}$$

$$+ \left[c_k^{(1)}\right]^2 P_k^{(1)} + 2c_k^{(1)} c_k^{(2)} P_k^{(12)} + \left[c_k^{(2)}\right]^2 P_k^{(2)}.$$

or

$$P_k^{GMF} = \frac{r_1 r_2 \sigma_\theta^2}{\left(r_1 + r_2\right)^2}\left[\frac{r_2}{r_1 + k\sigma_\theta^2}\right.$$

$$\left. + \frac{2 r_1 r_2}{\left(r + k\sigma_\theta^2\right)\left(r + k\sigma_\theta^2\right)} + \frac{r_1}{r_2 + k\sigma_\theta^2}\right]. \qquad (33)$$

Comparing the error variances (27) and (33), we have

$$P_k^{GMF} - P_k^{KF} = \frac{r_1^2 r_2^2 \sigma_\theta^4}{\left(r_1 + r_2\right)} \times$$

$$\times \frac{k}{\left(r_1 + k\sigma_\theta^2\right)\left(r_2 + k\sigma_\theta^2\right)\left[r_1 r_2 + k\left(r_1 + r_2\right)\sigma_\theta^2\right]}$$

$$= O\left(\frac{1}{k^2}\right).$$

At the values of parameters

$r_1 = 1$, $r_2 = 2$ and $\sigma_\theta^2 = 1$,

we have

$$P_k^{GMF} - P_k^{KF} \cong 0.44 \times \frac{1}{k^2}.$$

The result show that the GMF yields suboptimal recursive filter with good accuracy and certain well-defined convergence properties. It provides the best balance between computational
efficiency and desired estimation accuracy.

6. CONCLUSION

In this paper we present the GMF, which represents the optimal linear combination of arbitrary number correlated estimates. Each estimate is fused by the minimum mean square error criterion. Based on the GMF new suboptimal adaptive filters have derived. These filters have parallel structure and are very suitable for parallel processing of measurements. The obtained filtering algorithms reduce the computational burden and on-line computational requirements. The example demonstrate the efficiency and high-accuracy of the proposed filtering algorithms.

These filters can be widely used in the different areas of applications: industrial, military, space, communication, target tracking, inertial navigation and others.

REFERENCES

[1] F.L. Lewis, *Optimal Estimation with an Introduction to Stochastic Control Theory*, John Wiley & Sons, New York, 1986.

[2] Y. Bar-Shalom and L. Campo, The effect of the common process noise on the two-sensor fused-track covariance, *IEEE Trans. Aerospace and Electronic Systems*, Vol. 22, No. 11, pp. 803-805, 1986.

[3] Y. Bar-Shalom, X. Rong Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*, John Wiley & Sons, New York, 2001.

[4] D.G. Lainiotis, Partitioned linear estimation algorithm: discrete case, *IEEE Trans. Automat. Contro*l, Vol. 20, No. 3, pp. 255-257, 1975.

[5] W. Watanabe, *Adaptive Estimation and Control*: *Partitioning Approach*, Prentice-Hall, New York, 1991.

[6] A. Gelb, *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974.

[7] C.L. Ren and M.G. Kay, Multisensor Integration and Fusion in Intelligent Systems, *IEEE Trans. Systems, Man, and Cybernetics*, Vol. 19, No. 5, pp. 901-931, 1989.